

# $\ell^p(\mathbb{Z}^d)$ -ESTIMATES FOR DISCRETE OPERATORS OF RADON TYPE: MAXIMAL FUNCTIONS AND VECTOR-VALUED ESTIMATES

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ABSTRACT. We show  $\ell^p(\mathbb{Z}^d)$  boundedness, for  $p \in (1, \infty)$ , of discrete maximal functions corresponding to averaging operators and truncated singular integrals of Radon type. We shall present a new approach which allows us to handle these operators in a unified way. Our methods will be robust enough to provide vector-valued estimates for these maximal functions as well.

## 1. INTRODUCTION

The aim of this paper is to study discrete analogues of operators of Radon type. A wide class of interesting questions at the interface of harmonic analysis, number theory and ergodic theory that arise when discrete averaging operators or singular integrals modeled on polynomial mappings are studied. The approach undertaken in this paper has the merit of unifying results of two major streams of the theory, the Bourgain maximal theorems, and the theorem of Ionescu–Wainger for singular Radon transforms. In particular, we obtain analogous results for the maximal truncated singular Radon transforms, and for both not only in the scalar-valued case but also in a vector-valued version.

To begin, assume that  $K \in \mathcal{C}^1(\mathbb{R}^k \setminus \{0\})$  is a Calderón–Zygmund kernel satisfying the differential inequality

$$(1.1) \quad |y|^k |K(y)| + |y|^{k+1} |\nabla K(y)| \leq 1$$

for all  $y \in \mathbb{R}^k$  with  $|y| \geq 1$  and the cancellation condition

$$(1.2) \quad \sup_{\lambda \geq 1} \left| \int_{1 \leq |y| \leq \lambda} K(y) \, dy \right| \leq 1.$$

Let

$$\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{d_0}) : \mathbb{Z}^k \rightarrow \mathbb{Z}^{d_0}$$

be a polynomial mapping where for each  $j \in \{1, \dots, d_0\}$  the function  $\mathcal{P}_j : \mathbb{Z}^k \rightarrow \mathbb{Z}$  is an integer-valued polynomial of  $k$  variables with  $\mathcal{P}_j(0) = 0$ . We are interested in discrete truncated singular Radon transforms

$$T_N^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{Z}_N^k \setminus \{0\}} f(x - \mathcal{P}(y)) K(y)$$

defined for a finitely supported function  $f : \mathbb{Z}^{d_0} \rightarrow \mathbb{C}$ , where  $\mathbb{Z}_N^k = \{-N, \dots, -1, 0, 1, \dots, N\}^k$ .

Our starting was the proof of the following theorem.

**Theorem A.** *For every  $p \in (1, \infty)$  there is  $C_p > 0$  such that for all  $f \in \ell^p(\mathbb{Z}^{d_0})$*

$$\left\| \sup_{N \in \mathbb{N}} |T_N^{\mathcal{P}} f| \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

*Moreover, the constant  $C_p$  is independent of the coefficients of the polynomial mapping  $\mathcal{P}$ .*

Theorem A generalizes recent result of Ionescu and Wainger [10], where it was proven that the discrete singular Radon transform

$$T^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{Z}^k \setminus \{0\}} f(x - \mathcal{P}(y)) K(y)$$

is bounded on  $\ell^p(\mathbb{Z}^{d_0})$  for any  $p \in (1, \infty)$ .

Once this theorem was proved, it became clear that the approach used gave a different proof of the Bourgain theorem, and ultimately the vector-valued versions given in Theorems B and C below.

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The Bourgain-type theorems deal with the Radon averaging operators

$$(1.3) \quad M_N^{\mathcal{P}} f(x) = N^{-k} \sum_{y \in \mathbb{N}_N^k} f(x - \mathcal{P}(y))$$

defined for any finitely supported function  $f : \mathbb{Z}^{d_0} \rightarrow \mathbb{C}$ , where  $\mathbb{N}_N^k = \{1, 2, \dots, N\}^k$ .

Bourgain, being motivated by some questions in the pointwise ergodic theory, studied intensively the maximal Radon transforms associated with  $M_N^{\mathcal{P}} f$ . Specifically, in [3] it was proven, in one dimensional case with  $k = d_0 = 1$ , that  $\sup_{N \in \mathbb{N}} |M_N^{\mathcal{P}} f|$  is bounded on  $\ell^p(\mathbb{Z})$  for any  $1 < p \leq \infty$ . For higher dimensional cases with general  $k \geq 1$  and  $d_0 \geq 1$  we refer to [13], quasi-invariant analogues of (1.3), with polynomials of degree at most two, were also considered in [9].

We extend Bourgain's celebrated theorems from [1], [2] and [3] and provide some vector-valued estimates for the sequence  $(M_N^{\mathcal{P}} f : N \in \mathbb{N})$ . Let

$$\ell^p(\ell^2(\mathbb{Z}^{d_0})) = \left\{ (f_t : t \in \mathbb{N}) : \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p} < \infty \right\}.$$

The main results of this article will be Theorem B and Theorem C.

**Theorem B.** *For every  $p \in (1, \infty)$  there is  $C_p > 0$  such that for all  $(f_t : t \in \mathbb{N}) \in \ell^p(\ell^2(\mathbb{Z}^{d_0}))$*

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{N \in \mathbb{N}} |M_N^{\mathcal{P}} f_t|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

Moreover, the constant  $C_p$  is independent of the coefficients of the polynomial mapping  $\mathcal{P}$ .

In fact Theorem A will be a corollary of the following vector-valued estimates.

**Theorem C.** *For every  $p \in (1, \infty)$  there is  $C_p > 0$  such that for all  $(f_t : t \in \mathbb{N}) \in \ell^p(\ell^2(\mathbb{Z}^{d_0}))$*

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{N \in \mathbb{N}} |T_N^{\mathcal{P}} f_t|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

Moreover, the constant  $C_p$  is independent of the coefficients of the polynomial mapping  $\mathcal{P}$ .

The  $r$ -variation versions of these theorems will be the subject of the second paper of this series [12]. We next want to highlight three main tools that are used:

- (i) A variant of a key idea of Ionescu–Wainger [10]: a suitable  $\ell^p$  estimate for projection operators which are made up of sums corresponding to fractions  $a/q$  whose denominators  $q$  have appropriate limitation in terms of their prime power factorization (see (1.4) and inequality (1.5) below as well as Theorem 5.1).
- (ii) A maximal estimate in terms of “dyadic” sub-blocks (see (1.8)). It is a consequence of a numerical maximal estimate (see Lemma 2.2), which in turn is an outgrowth of the idea implicit in the proof of the classical Rademacher–Menshov theorem (see [18]).
- (iii) A refinement of the estimates for multi-dimensional Weyl sums in [16], where the previous limitations  $N^\epsilon \leq q \leq N^{k-\epsilon}$  are replaced by the weaker restrictions  $(\log N)^\beta \leq q \leq N^k (\log N)^{-\beta}$  for suitable  $\beta$ .

The proof of Theorem B and Theorem C will be based on analysis of the Fourier multipliers corresponding to operators  $M_{2^n}^{\mathcal{P}}$  and  $T_{2^n}^{\mathcal{P}}$  respectively. In order to describe the key points of our approach let us focus our attention on  $m_{2^n}$  which is the multiplier associated with  $M_{2^n}^{\mathcal{P}}$ , i.e.  $\mathcal{F}^{-1}(m_{2^n} \hat{f}) = M_{2^n}^{\mathcal{P}} f$ . Assume, for simplicity, that we are in the scalar case, furthermore,  $k = 1$  and our polynomial mapping is a moment curve  $\mathcal{P}(x) = (x^d, x^{d-1}, \dots, x)$  for some  $d_0 = d \geq 2$ . As in the previous papers in the subject we will employ the circle method of Hardy and Littlewood. These techniques will be used implicitly in analysis of relevant partition of unity which allows us to distinguish between asymptotic or highly oscillatory behaviour of  $m_{2^n}$ . In the asymptotic part one approximates the multiplier by sums corresponding to the continuous analogue, multiplied by Gauss sums. The highly oscillatory part is controlled by Weyl sums. In particular, let  $\eta$  be a smooth cut-off function with a small support, fix  $l \in \mathbb{N}$  and define for each  $n \in \mathbb{N}$  the following projections

$$(1.4) \quad \Xi_n(\xi) = \sum_{a/q \in \mathcal{U}_{n^l}} \eta(\mathcal{E}_n^{-1}(\xi - a/q))$$

where  $\mathcal{E}_n$  is a diagonal  $d \times d$  matrix with positive entries  $(\varepsilon_j : 1 \leq j \leq d)$  such that  $\varepsilon_j \leq e^{-n^{1/5}}$  and

$$\mathcal{U}_{n^l} = \{a/q \in \mathbb{T}^d \cap \mathbb{Q}^d : a = (a_1, \dots, a_d) \in \mathbb{N}_q^d \text{ and } \gcd(a_1, \dots, a_d, q) = 1 \text{ and } q \in P_{n^l}\}$$

for some family  $P_{n^l}$  such that  $\mathbb{N}_{n^l} \subseteq P_{n^l} \subseteq \mathbb{N}_{e^{n^{1/10}}}$ , we refer to Section 5 for detailed constructions. The projections  $\Xi_n$  will play an essential role in our further studies of  $m_{2^n}$ . On the one hand, following the ideas of Ionescu and Wainger [10], we will be able to prove that for every  $p \in (1, \infty)$  there is a constant  $C_{l,p} > 0$  such that

$$(1.5) \quad \|\mathcal{F}^{-1}(\Xi_n \hat{f})\|_{\ell^p} \leq C_{l,p} \log(n+2) \|f\|_{\ell^p}.$$

On the other hand,  $\Xi_n(\xi)m_{2^n}(\xi)$  localizes asymptotic behaviour of  $m_{2^n}(\xi)$ , whereas  $(1 - \Xi_n(\xi))m_{2^n}(\xi)$  corresponds to the highly oscillatory part. The last piece in view of a variant of Weyl's inequality with logarithmic decay, see Theorem 3.1, can be nicely controlled giving

$$\left\| \sup_{n \in \mathbb{N}} |\mathcal{F}^{-1}(m_{2^n}(1 - \Xi_n)\hat{f})| \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

The first part requires more sophisticated investigations. Namely, for every  $a/q \in \mathcal{U}_{n^l}$

$$m_{2^n}(\xi) = G(a/q)\Phi_{2^n}(\xi - a/q) + \mathcal{O}(2^{-n/2})$$

where  $G(a/q)$  is the Gaussian sum and  $\Phi_{2^n}$  is an integral version of  $m_{2^n}$ , see at the end of Section 3 and the beginning of Section 6 respectively for relevant definitions. This observation allows us to reduce the problem to showing that for each  $s \geq 0$  the maximal function associated with the multiplier

$$(1.6) \quad m_{2^n}^s(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)l} \setminus \mathcal{U}_{s,l}} G(a/q)\Phi_{2^n}(\xi - a/q)\eta(\mathcal{E}_s^{-1}(\xi - a/q))$$

is bounded on  $\ell^p(\mathbb{Z}^d)$  and has a decay given by

$$(1.7) \quad \left\| \sup_{n \in \mathbb{N}} |\mathcal{F}^{-1}(m_{2^n}^s \hat{f})| \right\|_{\ell^p} \leq C_p (s+1)^{-2} \|f\|_{\ell^p}.$$

The last bound is summable in  $s \geq 0$  and so if we prove that we will be done. The idea lying behind the estimates for (1.7) is based on proving  $\ell^2(\mathbb{Z}^d)$  estimates with bound  $(s+1)^{-\delta l+1} \|f\|_{\ell^2}$ , where  $\delta > 0$  is an exponent from the bound for the Gaussian sums  $|G(a/q)| \leq Cq^{-\delta}$ . For general  $p \neq 2$  we obtain much worse bound  $C_{l,p}s \log(s+2) \|f\|_{\ell^p}$ , but still satisfactory to obtain the desired conclusion from (1.7), since  $l \in \mathbb{N}$  can be arbitrary large and interpolation does the job. Technically, in either case ( $\ell^2(\mathbb{Z}^d)$  case Theorem 6.1 or  $\ell^p(\mathbb{Z}^d)$  case for  $p \neq 2$  Theorem 6.2 and Theorem 6.4) this will be achieved by partitioning the supremum in (1.7) into two pieces  $n \leq 2^{\kappa_s}$  and  $n > 2^{\kappa_s}$  for some integer  $1 < \kappa_s \leq Cs$ . The second case is quite straightforward and follows by appealing to the transference principle described in Section 4, where discrete  $\|\cdot\|_{\ell^p}$  norm of the maximal function associated with multiplier from (1.6) is controlled by the continuous  $\|\cdot\|_{L^p}$  norm of a maximal function closely related with the multiplier  $\Phi_{2^n}$ , which is *a priori* bounded on  $\mathbb{R}^d$ , see Theorem 6.4. The first case where the supremum is restricted to  $n \leq 2^{\kappa_s}$  is much more challenging, see Theorem 6.2. For this purpose we shall apply a simple numerical inequality, see Lemma 2.2, which ensures

$$(1.8) \quad \sup_{0 \leq n \leq 2^{\kappa_s}} |\mathcal{F}^{-1}(m_{2^n}^s \hat{f})| \leq |\mathcal{F}^{-1}(m_1^s \hat{f})| + \sqrt{2} \sum_{i=0}^{\kappa_s} \left( \sum_{j=0}^{2^{\kappa_s-i}-1} |\mathcal{F}^{-1}(m_{2^{(j+1)2^i}}^s \hat{f}) - \mathcal{F}^{-1}(m_{2^{j2^i}}^s \hat{f})|^2 \right)^{1/2}.$$

The advantage of application Lemma 2.2 is that the supremum norm is controlled by a sum of square functions which is more convenient to deal with. Following now the ideas of Ionescu and Wainger [10] via Khintchine's inequality we will be able to show that  $\ell^p(\mathbb{Z}^d)$  norm of the inner square function on the right-hand side in (1.8) is bounded by  $C_p \log(s+2) \|f\|_{\ell^p}$  for each  $0 \leq i \leq \kappa_s$ . Finally there are  $\kappa_s + 1$  elements thus the left hand side of (1.7) with the supremum restricted to  $n \leq 2^{\kappa_s}$  is bounded by  $C_{l,p}s \log(s+2) \|f\|_{\ell^p}$  for all  $p \in (1, \infty)$  and we are done. Loosely speaking this illustrates a general scheme which lies behind the proofs of all theorems in this paper. One of the most important aspects of this approach is that it allows us to treat with maximal functions associated with operators with signs and we can do it, unlike the previous approaches, in a fairly unified way.

We describe now more precisely the outline of this paper and ingredients in the proofs of Theorem B and Theorem C. In Section 2 we introduce a lifting procedure (see Lemma 2.1) which allows us to replace any polynomial mapping  $\mathcal{P}$  by the canonical polynomial mapping  $\mathcal{Q}$  which has all coefficients equal to 1. As a result our bounds will be independent of coefficients of the underlying polynomial mapping. The rest of this section is devoted to proving Lemma 2.2 which is critical in bounding the supremum norm by square functions, as we have seen in (1.8). In [13] it has been recently shown that Lemma 2.2 can be treated as a counterpart or replacement of Bourgain's logarithmic lemma [3, Lemma 4.1, page 17], which was essential tool in analysis of maximal functions corresponding to Radon averages. Although logarithmic

lemma found many applications in harmonic analysis, especially in the time-frequency analysis, in the discrete harmonic analysis turned out to be limited to averaging operators. Fortunately, Lemma 2.2 is more flexible and permits to circumvent all difficulties while operators with signs are considered. It is worth emphasizing that Lemma 2.2 is an invaluable ingredient allowing for the unification in the discrete analogues of operators of Radon type.

In Section 3 we present the variant of multidimensional Weyl's sum estimates with logarithmic decay, see Theorem 3.1. It was known that for Weyl's sums  $S_N$  defined at the beginning of Section 3 that  $|S_N| \leq CN^{k-\delta}$  for some  $\delta > 0$  provided that for at least one coefficient  $\xi_{\gamma_0}$  of a phase polynomial  $P$  there are integers  $a$  and  $q$  such that  $(a, q) = 1$ ,  $|\xi_{\gamma_0} - a/q| \leq q^{-2}$  and  $N^\varepsilon \leq q \leq N^{|\gamma_0|-\varepsilon}$  for some  $\varepsilon > 0$ . In the sequel proceeding as in [16] we will be able to prove that  $|S_N| \leq CN^k(\log N)^{-\alpha}$  with arbitrary large  $\alpha > 0$  provided that  $(\log N)^\beta \leq q \leq N^{|\gamma_0|}(\log N)^{-\beta}$  for some large  $\beta > 0$ . This logarithmic decay has great importance for our further analysis of multipliers in highly oscillatory regime. A one dimensional variant of Weyl's inequality with logarithmic decay is known in number theory for a long time, see for instance [19] and the references therein.

In Section 4 we gather some basic tools which allow us to efficiently compare discrete  $\|\cdot\|_{\ell^p}$  norms with continuous  $\|\cdot\|_{L^p}$  norms.

In Section 5 we prove Theorem 5.1 which is a key ingredient in the all steps of our proofs, especially in the proof of boundedness (1.5). This theorem was proven by Ionescu and Wainger in [10] with  $(\log N)^D$  loss where  $D > 0$  is a large power. This is a deep result which uses the most sophisticated tools developed to date in the area of discrete analogues in harmonic analysis. The main new idea of Ionescu and Wainger uses Diophantine approximation of the coefficients of the underlying polynomial like in the circle method, but unlike previous arguments, the authors exploit strongly their strong almost orthogonality properties. Here we improve their result and show that  $D = 1$  is sufficient. Moreover  $\log N$  loss is sharp for the method of proof which will be used. We also present a slightly different approach, based on properties of some square functions, see (5.12), Lemma 5.5 and Lemma 5.6, which explains the nature of strong orthogonality properties between distinct rational fractions.

Sections 6 and Section 7 complete the proofs of Theorems B and C respectively. To understand more quickly the structure of the proofs, the reader may begin by looking at Sections 6 and 7 first. These sections can be read independently, assuming the results in the previous sections.

In the Appendix A, which is self-contained, we collect vector-valued estimates for the maximal functions of Radon type in the continuous settings. The proofs of Theorem A.1 and Theorem A.2 can be found in [14]. However, we provided short proofs of these results and for the convenience of the reader we present the details.

Finally, let us comment that Theorem A also finds its applications in pointwise ergodic theory. Namely, let  $(X, \mathcal{B}, \mu)$  be a  $\sigma$ -finite measure space with a family of invertible commuting and measure preserving transformations  $S_1, S_2, \dots, S_{d_0}$ . Let

$$\mathcal{T}_N^P f(x) = \sum_{y \in \mathbb{B}_N \setminus \{0\}} f(S_1^{P_1(y)} S_2^{P_2(y)} \dots S_{d_0}^{P_{d_0}(y)} x) K(y)$$

where  $\mathbb{B}_N = \{x \in \mathbb{Z}^k : |x| \leq N\}$ . Here instead of condition (1.2) we need somewhat stronger condition

$$\int_{B_t \setminus B_{t'}} K(y) dy = 0$$

for every  $0 < t' \leq t$ , where  $B_t$  is the Euclidean ball in  $\mathbb{R}^k$  with radius  $t > 0$  centered at the origin. Then we can establish the following

**Theorem D.** *Assume that  $p \in (1, \infty)$ . For every  $f \in L^p(X, \mu)$  there exists  $f^* \in L^p(X, \mu)$  such that*

$$\lim_{N \rightarrow \infty} \mathcal{T}_N^P f(x) = f^*(x)$$

*$\mu$ -almost everywhere on  $X$ .*

Theorem D can be treated as an extension of Cotlar's ergodic theorem [5]. The proof of Theorem D follows in view of the Calderón transference principle from Theorem A and estimates of oscillation inequality on  $L^2(X, \mu)$  for  $\mathcal{T}_N^P$ . However we will not present the details, since we are going to provide much more general  $r$ -variational estimates for  $M_N^P$  and  $T_N^P$  in [12], and then the pointwise convergence will immediately follow.

There is also an ongoing project to obtain Theorems A, B and C for discrete operators of Radon type modeled on thin subsets of integers, for example prime numbers.

**1.1. Notation.** Throughout the whole article, unless otherwise stated, we will write  $A \lesssim B$  ( $A \gtrsim B$ ) if there is an absolute constant  $C > 0$  such that  $A \leq CB$  ( $A \geq CB$ ). Moreover,  $C > 0$  will stand for a large positive constant whose value may vary from occurrence to occurrence. If  $A \lesssim B$  and  $A \gtrsim B$  hold simultaneously then we will write  $A \simeq B$ . We will also write  $A \lesssim_\delta B$  ( $A \gtrsim_\delta B$ ) to indicate that the constant  $C > 0$  depends on some  $\delta > 0$ . Let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . For  $N \in \mathbb{N}$  we set

$$\mathbb{N}_N = \{1, 2, \dots, N\}, \quad \text{and} \quad \mathbb{Z}_N = \{-N, \dots, -1, 0, 1, \dots, N\}.$$

For a vector  $x \in \mathbb{R}^d$  we will use the following norms

$$|x|_\infty = \max\{|x_j| : 1 \leq j \leq d\}, \quad \text{and} \quad |x| = \left( \sum_{j=1}^d |x_j|^2 \right)^{1/2}.$$

If  $\gamma$  is a multi-index from  $\mathbb{N}_0^k$  then  $|\gamma| = \gamma_1 + \dots + \gamma_k$ . Although we use  $|\cdot|$  for the length of a multi-index  $\gamma \in \mathbb{N}_0^k$  and the Euclidean norm of  $x \in \mathbb{R}^d$ , their meaning will be always clear from the context and it will cause no confusions in the sequel. Finally, let  $\mathcal{D} = \{2^n : n \in \mathbb{N}_0\}$  denote the set of dyadic numbers.

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## 2. PRELIMINARIES

**2.1. Lifting lemma.** Let  $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{d_0}) : \mathbb{Z}^k \rightarrow \mathbb{Z}^{d_0}$  be a polynomial mapping whose components  $\mathcal{P}_j$  are integer valued polynomials on  $\mathbb{Z}^k$  such that  $\mathcal{P}_j(0) = 0$ . We set

$$N_0 = \max\{\deg \mathcal{P}_j : 1 \leq j \leq d_0\}.$$

It is convenient to work with the set

$$\Gamma = \{\gamma \in \mathbb{Z}^k \setminus \{0\} : 0 \leq \gamma_j \leq N_0 \text{ for each } j = 1, \dots, k\}$$

with the lexicographic order. Then each  $\mathcal{P}_j$  can be expressed as

$$\mathcal{P}_j(x) = \sum_{\gamma \in \Gamma} c_j^\gamma x^\gamma$$

for some  $c_j^\gamma \in \mathbb{Z}$ . Let us denote by  $d$  the cardinality of the set  $\Gamma$ . We identify  $\mathbb{R}^d$  with the space of all vectors whose coordinates are labeled by multi-indices  $\gamma \in \Gamma$ . Let  $A$  be a diagonal  $d \times d$  matrix such that

$$(Av)_\gamma = |\gamma|v_\gamma.$$

For  $t > 0$  we set

$$t^A = \exp(A \log t)$$

i.e.  $t^A x = (t^{|\gamma|} x_\gamma : \gamma \in \Gamma)$  for every  $x \in \mathbb{R}^d$ . Next, we introduce the *canonical* polynomial mapping

$$\mathcal{Q} = (\mathcal{Q}_\gamma : \gamma \in \Gamma) : \mathbb{Z}^k \rightarrow \mathbb{Z}^d$$

where  $\mathcal{Q}_\gamma(x) = x^\gamma$  and  $x^\gamma = x_1^{\gamma_1} \dots x_k^{\gamma_k}$ . The coefficients  $(c_j^\gamma : \gamma \in \Gamma, j \in \{1, \dots, d_0\})$  define a linear transformation  $L : \mathbb{R}^d \rightarrow \mathbb{R}^{d_0}$  such that  $L\mathcal{Q} = \mathcal{P}$ . Indeed, it is enough to set

$$(Lv)_j = \sum_{\gamma \in \Gamma} c_j^\gamma v_\gamma$$

for each  $j \in \{1, \dots, d_0\}$  and  $v \in \mathbb{R}^d$ .

Let  $\mathcal{N}$  denote a seminorm defined on sequences of complex numbers, i.e. a non-negative function such that for any two sequences  $(a_j : j \in J)$  and  $(b_j : j \in J)$  where  $J \subseteq \mathbb{Z}$  satisfies

$$\mathcal{N}(a_j + b_j : j \in J) \leq \mathcal{N}(a_j : j \in J) + \mathcal{N}(b_j : j \in J),$$

and for any  $\lambda \in \mathbb{C}$

$$\mathcal{N}(\lambda a_j : j \in J) = |\lambda| \mathcal{N}(a_j : j \in J).$$

Moreover, we assume that

$$\mathcal{N}(a_j : j \in J) \leq \left( \sum_{j \in J} |a_j|^2 \right)^{1/2}$$

and for any  $J_1 \subseteq J_2 \subseteq \mathbb{Z}$  we have

$$\mathcal{N}(a_j : j \in J_1) \leq \mathcal{N}(a_j : j \in J_2).$$

For the future reference the tools presented in this and further sections are stated and proved in higher generality than needed. For the first reading one may think that

$$\mathcal{N}(a_j : j \in J) = \sup_{j \in J} |a_j|.$$

Let  $(X, \mu)$  be a measurable space and fix  $\mathcal{Z} \subseteq \mathbb{N}$  and  $p, r \in [1, \infty]$ . We say that a sequence of complex-valued functions  $(f_t : t \in \mathcal{Z}) \in L^p(\ell_{\mathcal{Z}}^r(X))$  if

$$\left\| \left( \sum_{t \in \mathcal{Z}} |f_t|^r \right)^{1/r} \right\|_{L^p(X)} < \infty$$

with obvious modifications when  $p = \infty$  or  $r = \infty$ . We will write  $L^p(\ell_{\mathcal{Z}}^r(X)) = L^p(\ell^r(X))$  when  $\mathcal{Z} = \mathbb{N}$  and  $L^p(\ell_{\mathcal{Z}}^r(X)) = L^p(X)$  when  $\mathcal{Z} = \{1\}$ . In our case  $(X, \mu)$  will be usually  $\mathbb{R}^d$  with the Lebesgue measure or  $\mathbb{Z}^d$  with the counting measure.

The next lemma, inspired by the continuous analogue (see [7] or [15, Section 11]) reduces the proof of Theorem A to the canonical polynomial mapping.

**Lemma 2.1.** *Let  $R_N^{\mathcal{P}}$  be one of the operators  $M_N^{\mathcal{P}}$  or  $T_N^{\mathcal{P}}$ . Suppose that for some  $p, r \in [1, \infty]$  there is a constant  $C = C_{p,r} > 0$  such that*

$$(2.1) \quad \left\| \left( \sum_{t \in \mathcal{Z}} \mathcal{N}(R_N^{\mathcal{Q}} f_t : N \in \mathbb{N})^r \right)^{1/r} \right\|_{\ell^p(\mathbb{Z}^d)} \leq C \left\| \left( \sum_{t \in \mathcal{Z}} |f_t|^r \right)^{1/r} \right\|_{\ell^p(\mathbb{Z}^d)}.$$

Then

$$(2.2) \quad \left\| \left( \sum_{t \in \mathcal{Z}} \mathcal{N}(R_N^{\mathcal{P}} f_t : N \in \mathbb{N})^r \right)^{1/r} \right\|_{\ell^p(\mathbb{Z}^{d_0})} \leq C \left\| \left( \sum_{t \in \mathcal{Z}} |f_t|^r \right)^{1/r} \right\|_{\ell^p(\mathbb{Z}^{d_0})}.$$

*Proof.* Let  $M > 0$  and  $\Lambda > 0$  be fixed. Let  $f \in \ell^p(\mathbb{Z}^{d_0})$ . In the proof we let  $x \in \mathbb{Z}^{d_0}$ ,  $y \in \mathbb{Z}^k$  and  $u \in \mathbb{Z}^d$ . For any  $x \in \mathbb{Z}^{d_0}$  we define a function  $F_t^x$  on  $\mathbb{Z}^d$  by

$$F_t^x(z) = \begin{cases} f_t(x + L(z)) & \text{if } |z|_{\infty} \leq M + \Lambda^{kN_0}, \\ 0 & \text{otherwise.} \end{cases}$$

If  $|y|_{\infty} \leq N \leq \Lambda$  and  $|u|_{\infty} \leq M$  then  $|u - \mathcal{Q}(y)|_{\infty} \leq M + \Lambda^{kN_0}$ . Therefore for each  $x \in \mathbb{Z}^{d_0}$

$$R_N^{\mathcal{P}} f_t(x + Lu) = \sum_{y \in \mathbb{Z}_N^k \setminus \{0\}} f_t(x + L(u - \mathcal{Q}(y))) H_N(y) = R_N^{\mathcal{Q}} F_t^x(u),$$

where

$$H_N(y) = \begin{cases} N^{-k} \mathbf{1}_{[1, N]^k}(y) & \text{if } R_N^{\mathcal{P}} = M_N^{\mathcal{P}}, \\ \mathbf{1}_{[-N, N]^k \setminus \{0\}}(y) K(y) & \text{if } R_N^{\mathcal{P}} = T_N^{\mathcal{P}}. \end{cases}$$

Hence,

$$\begin{aligned} & \left\| \left( \sum_{t \in \mathcal{Z}} \mathcal{N}(R_N^{\mathcal{P}} f_t : N \in [1, \Lambda])^r \right)^{1/r} \right\|_{\ell^p(\mathbb{Z}^{d_0})}^p \\ &= \frac{1}{(2M+1)^d} \sum_{x \in \mathbb{Z}^{d_0}} \sum_{|u|_{\infty} \leq M} \left( \sum_{t \in \mathcal{Z}} \mathcal{N}(R_N^{\mathcal{P}} f_t(x + Lu) : N \in [1, \Lambda])^r \right)^{p/r} \\ &= \frac{1}{(2M+1)^d} \sum_{x \in \mathbb{Z}^{d_0}} \sum_{|u|_{\infty} \leq M} \left( \sum_{t \in \mathcal{Z}} \mathcal{N}(R_N^{\mathcal{Q}} F_t^x(u) : N \in [1, \Lambda])^r \right)^{p/r} \\ &\leq \frac{C^p}{(2M+1)^d} \sum_{x \in \mathbb{Z}^{d_0}} \sum_{u \in \mathbb{Z}^d} \left( \sum_{t \in \mathcal{Z}} |F_t^x(u)|^r \right)^{p/r} \end{aligned}$$

where in the last inequality we have used (2.1). Since

$$\begin{aligned} \sum_{x \in \mathbb{Z}^{d_0}} \sum_{u \in \mathbb{Z}^d} \left( \sum_{t \in \mathcal{Z}} |F_t^x(u)|^r \right)^{p/r} &= \sum_{x \in \mathbb{Z}^{d_0}} \sum_{|u|_\infty \leq M + \Lambda^{kN_0}} \left( \sum_{t \in \mathcal{Z}} |f_t(x + Lu)|^r \right)^{p/r} \\ &\leq (2M + 2\Lambda^{kN_0} + 1)^d \left\| \left( \sum_{t \in \mathcal{Z}} |f_t|^r \right)^{1/r} \right\|_{\ell^p(\mathbb{Z}^{d_0})}^p \end{aligned}$$

we get

$$\left\| \left( \sum_{t \in \mathcal{Z}} \mathcal{N}(R_N^{\mathcal{P}} f_t : N \in [1, \Lambda])^r \right)^{1/r} \right\|_{\ell^p(\mathbb{Z}^{d_0})}^p \leq C^p \left( 1 + \frac{\Lambda^{kN_0}}{M} \right)^d \left\| \left( \sum_{t \in \mathcal{Z}} |f_t|^r \right)^{1/r} \right\|_{\ell^p(\mathbb{Z}^{d_0})}^p.$$

Taking  $M$  approaching infinity we conclude

$$\left\| \left( \sum_{t \in \mathcal{Z}} \mathcal{N}(R_N^{\mathcal{P}} f_t : N \in [1, \Lambda])^r \right)^{1/r} \right\|_{\ell^p(\mathbb{Z}^{d_0})}^p \leq C^p \left\| \left( \sum_{t \in \mathcal{Z}} |f_t|^r \right)^{1/r} \right\|_{\ell^p(\mathbb{Z}^{d_0})}^p$$

which by monotone convergence theorem implies (2.2).  $\square$

In the rest of the article by  $M_N$  or  $T_N$  we denote the operator defined for the canonical polynomial mapping  $\mathcal{Q}$ , i.e.  $M_N = M_N^{\mathcal{Q}}$  and  $T_N = T_N^{\mathcal{Q}}$ .

**2.2. Basic lemma.** We finish this section with the following simple observation which will be essential in the sequel.

**Lemma 2.2.** *For any sequence  $(a_j : 0 \leq j \leq 2^s)$  of complex numbers we have*

$$(2.3) \quad \max_{0 \leq j \leq 2^s} |a_j| \leq |a_{j_0}| + \sqrt{2} \sum_{i=0}^s \left( \sum_{j=0}^{2^{s-i}-1} |a_{(j+1)2^i} - a_{j2^i}|^2 \right)^{1/2}$$

for any  $j_0 \in \{0, 1, \dots, 2^s\}$ .

*Proof.* We claim that any interval  $[m, n)$  for  $m, n \in \mathbb{N}_0$  such that  $0 \leq m < n \leq 2^s$ , is a finite disjoint union of dyadic subintervals, i.e. intervals belonging to some  $\mathcal{I}_i$  for  $0 \leq i \leq s$  where

$$\mathcal{I}_i = \{[j2^i, (j+1)2^i) : 0 \leq j \leq 2^{s-i} - 1\}$$

and such that each length appears at most twice. For the proof, we set  $m_0 = m$ . Having chosen  $m_p$  we select  $m_{p+1}$  in such a way that  $[m_p, m_{p+1})$  is the longest dyadic interval starting at  $m_p$  and contained inside  $[m_p, n)$ . If the lengths of the selected dyadic intervals increase then we are done. Otherwise, there is  $p$  such that  $m_{p+1} - m_p \geq m_{p+2} - m_{p+1}$ . We show that this implies  $m_{p+2} - m_{p+1} > m_{p+3} - m_{p+2}$ . Suppose for a contradiction that,  $m_{p+2} - m_{p+1} \leq m_{p+3} - m_{p+2}$ . Then

$$[m_{p+1}, 2m_{p+2} - m_{p+1}) \subseteq [m_{p+1}, m_{p+3}).$$

Therefore, it is enough to show that  $2(m_{p+2} - m_{p+1})$  divides  $m_{p+1}$ . It is clear in the case when  $m_{p+1} - m_p > m_{p+2} - m_{p+1}$ . If  $m_{p+1} - m_p = m_{p+2} - m_{p+1}$  then, by maximality of  $[m_p, m_{p+1})$ ,  $2(m_{p+2} - m_{p+1})$  cannot divide  $m_p$ , thus divides  $m_{p+1}$ .

Let us fix  $j_0 \in \{0, 1, \dots, 2^s\}$ . If  $j \in \{0, 1, \dots, j_0 - 1\}$  we may write

$$[j, j_0) = \bigcup_{p=0}^{P_j} [u_p^j, u_{p+1}^j)$$

for some  $P_j \geq 1$  and where each interval  $[u_p^j, u_{p+1}^j)$  is dyadic. Then

$$|a_{j_0} - a_j| \leq \sum_{p=0}^{P_j} |a_{u_{p+1}^j} - a_{u_p^j}| = \sum_{i=0}^s \sum_{p: [u_p^j, u_{p+1}^j) \in \mathcal{I}_i} |a_{u_{p+1}^j} - a_{u_p^j}|.$$

We notice that the inner sum contains at most two terms. Hence,

$$|a_j| \leq |a_{j_0}| + |a_{j_0} - a_j| \leq |a_{j_0}| + \sqrt{2} \sum_{i=0}^s \left( \sum_{p: [u_p^j, u_{p+1}^j) \in \mathcal{I}_i} |a_{u_{p+1}^j} - a_{u_p^j}|^2 \right)^{1/2}$$

which is bounded by the right-hand side of (2.3). Similar argument covers the case when  $j$  belongs to  $\{j_0 + 1, \dots, 2^s\}$ .  $\square$



## 3. EXPONENTIAL SUMS

Let  $P$  be a polynomial in  $\mathbb{R}^k$  of degree  $d \in \mathbb{N}$  such that

$$P(x) = P(x; \xi) = \sum_{0 < |\gamma| \leq d} \xi_\gamma x^\gamma.$$

Given  $N \geq 1$ , let  $\Omega_N$  be a convex set such that

$$\Omega_N \subseteq B_{cN}(x_0)$$

for some  $x_0 \in \mathbb{R}^k$  and  $c > 0$ , where  $B_r(x_0) = \{x \in \mathbb{R}^k : |x - x_0| \leq r\}$ . We define

$$S_N = S_N(\xi) = \sum_{n \in \Omega_N \cap \mathbb{Z}^k} e^{2\pi i P(n)} \varphi(n).$$

The function  $\varphi : \mathbb{R}^k \rightarrow \mathbb{C}$  is assumed to be a  $\mathcal{C}^1(\mathbb{R}^k)$  function which for some  $C > 0$  satisfies

$$(3.1) \quad |\varphi(x)| \leq C, \quad \text{and} \quad |\nabla \varphi(x)| \leq C(1 + |x|)^{-1}.$$

We are going to show the following theorem which is a refinement of [16, Proposition 3].

**Theorem 3.1.** *Assume that there is a multi-index  $\gamma_0$  such that  $0 < |\gamma_0| \leq d$  and*

$$\left| \xi_{\gamma_0} - \frac{a}{q} \right| \leq \frac{1}{q^2}$$

*for some integers  $a, q$  such that  $0 \leq a \leq q$  and  $(a, q) = 1$ . Then for any  $\alpha > 0$  there is  $\beta_\alpha > 0$  so that, for any  $\beta \geq \beta_\alpha$ , if*

$$(3.2) \quad (\log N)^\beta \leq q \leq N^{|\gamma_0|} (\log N)^{-\beta}$$

*then there is a constant  $C > 0$*

$$(3.3) \quad |S_N| \leq CN^k (\log N)^{-\alpha}.$$

*The implied constant  $C$  is independent of  $N$ .*

We need the following.

**Proposition 3.1.** *Fix  $0 \leq \sigma \leq 1/3$ . Suppose that  $\Omega \subseteq \mathbb{R}^k$  is a convex set contained in a ball with radius  $r \geq 1$ . If  $1 \leq s \leq r^{1-3\sigma}$ , then the number of lattice points  $N_\Omega$  in  $\Omega$  of distance  $< s$  from the boundary of  $\Omega$  is  $\mathcal{O}(sr^{k-1+2\sigma}) = \mathcal{O}(r^{k-\sigma})$ .*

*If we additionally assume that  $\Omega$  contains a ball  $B_{cr}(x'_0)$  for some  $x'_0 \in \mathbb{R}^k$  and  $c > 0$ , then the number of lattice points  $N_\Omega$  in  $\Omega$  of distance  $< s$  from the boundary of  $\Omega$  is  $\mathcal{O}(sr^{k-1})$  if  $1 \leq s \leq r$ .*

*Proof.* If  $|\Omega| \leq r^{k-\sigma}$ , then there is nothing to do since by Davenport's result [6] we know that

$$\#(\Omega \cap \mathbb{Z}^k) = |\Omega| + \mathcal{O}(r^{k-1})$$

where  $|\Omega|$  is the volume of  $\Omega$ . Assume that  $|\Omega| \geq r^{k-\sigma}$ . Then by a simple integration argument for each  $1 \leq j \leq k$  there is a segment  $I_j \subseteq \Omega$  in direction  $x_j$ , and  $|I_j| \geq \frac{|\Omega|}{r^{k-1}}$ . Thus there is a ball  $B_\rho(x'_0)$  with radius  $\rho = \frac{c|\Omega|}{r^{k-1}}$  for some  $c > 0$  contained in the convex set  $\Pi$  generated by all segments  $I_1, \dots, I_k$  with  $x'_0$  being the barycenter of  $\Pi$ . Since the number of lattice points is invariant under translations we may assume that  $x'_0 = 0$ . Observe that  $\rho \geq cr^{1-\sigma}$ .

For any  $x \in \Omega$ , let  $\bar{x} \in \partial\Omega$  so that  $x = \lambda_x \bar{x}$  with some  $\lambda_x \in (0, 1)$ . If  $\Gamma_{\bar{x}}$  is the convex set generated by  $\bar{x}$  and  $B_\rho(0)$ , then the angle of the aperture at the vertex  $\bar{x}$  is  $\geq \alpha$ , with some  $\alpha \geq \rho/r$  uniformly for  $x \in \Omega$ . Then there exists  $c' > 0$  such that for all  $x \in \Omega$  we get

$$\text{dist}(x, \partial\Omega) \geq \text{dist}(x, \partial\Gamma_{\bar{x}}) = (1 - \lambda_x)|\bar{x}| \sin(\alpha/2) \geq c'(1 - \lambda_x) \frac{\rho^2}{r}.$$

Define  $\delta = 1 - \frac{sr}{c'\rho^2}$  and observe that for every  $x \in \Omega_\delta = \{y \in \mathbb{R}^k : \delta^{-1}y \in \Omega\}$  we have

$$\text{dist}(x, \partial\Omega) \geq c'(1 - \lambda_x) \frac{\rho^2}{r} \geq c'(1 - \delta) \frac{\rho^2}{r} = s$$

which is equivalent to

$$\{x \in \Omega : \text{dist}(x, \partial\Omega) < s\} \subseteq \Omega \setminus \Omega_\delta.$$



Since  $\#(\Omega \cap \mathbb{Z}^k) = |\Omega| + \mathcal{O}(r^{k-1})$  and  $\#(\Omega_\delta \cap \mathbb{Z}^k) = \delta^k |\Omega| + \mathcal{O}(r^{k-1})$  so

$$\begin{aligned} N_\Omega &= \#\{x \in \Omega \cap \mathbb{Z}^k : \text{dist}(x, \partial\Omega) < s\} \leq \#(\Omega \cap \mathbb{Z}^k) - \#(\Omega_\delta \cap \mathbb{Z}^k) \\ &= \mathcal{O}((1-\delta)r^k) + \mathcal{O}(r^{k-1}) = \mathcal{O}(sr^{k-1+2\sigma}) + \mathcal{O}(r^{k-1}) = \mathcal{O}(r^{k-\sigma}). \end{aligned}$$

Suppose now that  $B_{cr}(x'_0) \subseteq \Omega \subseteq B_r(x_0)$  for some  $x_0, x'_0 \in \mathbb{R}^k$  and  $c > 0$ . Again without loss of generality we may assume that  $x'_0 = 0$ , and  $B_{cr}(x'_0)$  and  $\Omega$  are not tangent at any point, otherwise it suffices to take  $c/2$  instead of  $c$ . If  $\Gamma_{\bar{x}}$  is the convex set generated by  $\bar{x}$  and  $B_{cr}(0)$ , then the angle of the aperture at the vertex  $\bar{x}$  is  $\geq \alpha$ , with some  $\alpha = \alpha(c) > 0$  uniformly for  $x \in \Omega$ . Then there exists  $c' > 0$  such that for all  $x \in \Omega$  we get

$$\text{dist}(x, \partial\Omega) \geq \text{dist}(x, \partial\Gamma_{\bar{x}}) = (1 - \lambda_x)|\bar{x}| \sin(\alpha/2) \geq c'r(1 - \lambda_x).$$

Define  $\delta = 1 - \frac{s}{c'r}$  and observe that for every  $x \in \Omega_\delta = \{y \in \mathbb{R}^k : \delta^{-1}y \in \Omega\}$  we have

$$\text{dist}(x, \partial\Omega) \geq c'r(1 - \lambda_x) \geq c'r(1 - \delta) = s.$$

Thus  $\{x \in \Omega : \text{dist}(x, \partial\Omega) < s\} \subseteq \Omega \setminus \Omega_\delta$  and arguing in a similar way as above we obtain

$$N_\Omega = \mathcal{O}(sr^{k-1}).$$

The proof of Proposition 3.1 is completed.  $\square$

*Remark 3.1.* This proposition is needed as a replacement for [16, Proposition 9], since the latter proposition contains an error. While the present version is weaker than the one in [16], it is sufficiently strong for our purposes, and also to establish the Weyl sum estimates in [16].

*Proof of Theorem 3.1.*  $C, C', c, c', \dots$  will be constants that appear below. Their value will be adjusted several times, (but finitely many times) depending on the backward induction we use. The constants  $C, C', c, c', \dots$ , and the constants implicit in the  $\mathcal{O}$  notation, may depend on  $\alpha, \beta$ , the dimension  $k$ , the degree  $d$  of the polynomial  $P$ , but are independent of  $N$ .

Also sometimes we shall have the weaker hypothesis

$$C'(\log N)^\beta \leq q \leq \frac{1}{C'}N^{|\gamma_0|}(\log N)^{-\beta}$$

with  $C' < 1$  instead of (3.2). This can be reduced to (3.2) by taking  $\beta' < \beta$ , and using (3.2) with  $\beta'$  instead. Since

$$C'(\log N)^\beta > (\log N)^{\beta'}$$

except for finitely many  $N$ , and for those finitely many  $N$ , the result is subsumed if the constant  $C$  in (3.3) is taken sufficiently large.

Now the proof will proceed in four steps.

**Step 1.** Suppose  $k = 1$ ,  $|\gamma_0| = d$  and  $\varphi \equiv 1$ . We are reduced to proving that

$$(3.4) \quad |S'_{N'}| \leq CN(\log N)^{-(\alpha+1)}$$

for all  $N' \leq cN$ , where

$$S'_N = \sum_{n=1}^N e^{2\pi i P(n)}.$$

There are two cases  $N' \leq N(\log N)^{-(\alpha+1)}$  and  $N' \geq N(\log N)^{-(\alpha+1)}$ . The short sums are trivial because then

$$|S'_{N'}| = \mathcal{O}(N(\log N)^{-(\alpha+1)}).$$

It suffices to consider  $N' \geq N(\log N)^{-(\alpha+1)}$ . The desired bound in the second case follows by applying the Weyl estimates with logarithmic loss (see [19, Remark after Theorem 1.5]). Namely, if  $|\xi_d - a/q| \leq q^{-2}$  for some integers  $a, q$  such that  $0 \leq a \leq q$  and  $(a, q) = 1$  then there is  $C > 0$  such that

$$\left| \sum_{n=1}^N e^{2\pi i P(n)} \right| \leq C \log N \left( \frac{1}{q} + \frac{1}{N} + \frac{q}{N^d} \right)^{\frac{1}{2d^2-2d+1}}.$$

Therefore, for any  $\beta \geq \beta_\alpha = (\alpha + 2)(2d^2 - 2d + 1)$  if  $(\log N)^\beta \leq q \leq N^d(\log N)^{-\beta}$  we can immediately conclude

$$\left| \sum_{n=1}^N e^{2\pi i P(n)} \right| \leq C(\log N)^{1 - \frac{\beta}{2d^2-2d+1}} \leq C(\log N)^{-(\alpha+1)}.$$

We show next how to pass to a general function  $\varphi$  satisfying (3.1). Indeed, summing by parts we obtain

$$\sum_{n=1}^N e^{2\pi i P(n)} \varphi(n) = \sum_{n=1}^N (\varphi(n) - \varphi(n+1)) S'_n + \mathcal{O}(|S'_N|).$$

Therefore, we apply the conclusion of the special case to  $S'_n$ , and we are done.

**Step 2.** We assume that  $k = 1$ ,  $|\gamma_0| < d$ . We will argue by the backward induction on  $|\gamma_0|$ . We consider the case when  $|\gamma_0| = l \leq d - 1$  and assume that the desired bounds hold when  $|\gamma_0| = j$  and  $l < j \leq d$ . The result from Step 1 establishes the first step of the backward induction.

We write  $P(x) = \xi_d x^d + \xi_{d-1} x^{d-1} + \dots + \xi_l x^l + \text{lower degree terms}$ , and we assume that

$$(3.5) \quad \left| \xi_l - \frac{a}{q} \right| \leq \frac{1}{q^2}$$

for some integers  $a, q$  such that  $0 \leq a \leq q$  and  $(a, q) = 1$ , with

$$(\log N)^\beta \leq q \leq N^l (\log N)^{-\beta}.$$

We want to prove

$$(3.6) \quad |S_N| = \mathcal{O}(N(\log N)^{-\alpha}), \quad \text{where} \quad S_N = \sum_{n=1}^N e^{2\pi i P(n)} \varphi(n)$$

for  $\alpha > 0$ , if  $\beta > 0$  is large enough. We examine the coefficients  $\xi_d, \dots, \xi_{l+1}$  in the polynomial  $P$  and play with the dichotomy of major/minor arcs.

We fix  $\beta_1$ , to be determined later, and apply Dirichlet's principle, obtaining  $a_j/q_j$ , with  $(a_j, q_j) = 1$  and  $1 \leq q_j \leq N^j (\log N)^{-\beta_1}$ , so that

$$(3.7) \quad \left| \xi_j - \frac{a_j}{q_j} \right| \leq \frac{(\log N)^{\beta_1}}{q_j N^j}$$

for all  $l < j \leq d$ . There are two cases:

- *the minor arc case*, when for some  $l < j \leq d$  we have  $(\log N)^{\beta_1} \leq q_j \leq N^j (\log N)^{-\beta_1}$ ,
- *the major arc case*, when for all  $l < j \leq d$  we have  $1 \leq q_j \leq (\log N)^{\beta_1}$ .

In the first case we make the choice of  $\beta_1$  to be so large that the inductive hypothesis applies, giving us the conclusion (3.6) for the given  $\alpha > 0$ .

For the case when  $1 \leq q_j \leq (\log N)^{\beta_1}$  for all  $l < j \leq d$ , we need the following lemma.

**Lemma 3.1.** *Suppose*

$$\left| \theta - \frac{a}{q} \right| \leq \frac{1}{q^2}$$

and  $(\log N)^\beta \leq q \leq N^j (\log N)^{-\beta}$ . Let  $Q$  be an integer,  $Q \leq (\log N)^{\beta'}$  with  $\beta' < \beta$ . If

$$\beta_2 \leq \min\{\beta/2, \beta - \beta'\},$$

then there is  $a'/q'$  so that  $(a', q') = 1$  and

$$\left| Q\theta - \frac{a'}{q'} \right| \leq \frac{(\log N)^{\beta_2}}{q' N^j}$$

with  $(\log N)^{\beta_2} \leq q' \leq N^j (\log N)^{-\beta_2}$ .

Let us assume the lemma and see how it completes Step 2. We will take  $Q_1 = \text{lcm}(q_j : l < j \leq d)$ . Write  $\theta_j = \xi_j - a_j/q_j$ , for each  $l < j \leq d$  then (3.7) implies

$$(3.8) \quad |\theta_j| \leq \frac{(\log N)^{\beta_1}}{q_j N^j}$$

and  $Q_1 \leq (\log N)^{(d-l)\beta_1}$ . We then decompose  $\mathbb{Z}$  modulo  $Q_1$ , and write  $n = Q_1 m + r$ , with  $1 \leq r \leq Q_1$ . Thus

$$\sum_{n=1}^N e^{2\pi i P(n)} \varphi(n) = \sum_{r=1}^{Q_1} \sum_{m=1}^{\lfloor N/Q_1 \rfloor} e^{2\pi i P(Q_1 m + r)} \varphi(Q_1 m + r) + E$$

where  $|E| \leq (\log N)^{(d-l)\beta_1}$ , since  $E$  involves at most  $Q_1$  terms. Now,

$$P(Q_1 m + r) = \sum_{j=l+1}^d \xi_j (Q_1 m + r)^j + \xi_l (Q_1 m + r)^l + \text{a polynomial in } m \text{ of degree } \leq l-1.$$

Hence,

$$\begin{aligned} P(Q_1 m + r) &\equiv \sum_{j=l+1}^d (a_j/q_j)(Q_1 m + r)^j + \theta_j(Q_1 m + r)^j + \xi_l(Q_1 m + r)^l + \text{lower powers of } m \pmod{1} \\ &\equiv \sum_{j=l+1}^d (a_j/q_j)r^j + \sum_{j=l+1}^d \theta_j(Q_1 m + r)^j + \xi_l Q_1^l m^l + \text{lower powers of } m \pmod{1}. \end{aligned}$$

Thus

$$(3.9) \quad \sum_{n=1}^N e^{2\pi i P(n)} \varphi(n) = \sum_{r=1}^{Q_1} e^{2\pi i \sum_{j=l+1}^d (a_j/q_j) r^j} \sum_{m=1}^{\lfloor N/Q_1 \rfloor} A_{m,r} B_{m,r} + E$$

where

$$A_{m,r} = e^{2\pi i \sum_{j=l+1}^d \theta_j(Q_1 m + r)^j}, \quad B_{m,r} = e^{2\pi i (\xi_l Q_1^l m^l + R(m))} \varphi(Q_1 m + r),$$

with  $R$  being a polynomial in  $m$  of order  $\leq l-1$ . We estimate the inner sum in (3.9) by writing it as

$$\sum_{m=1}^{\lfloor N/Q_1 \rfloor} A_{m,r} B_{m,r} = \sum_{m=1}^{\lfloor N/Q_1 \rfloor} (A_{m,r} - A_{m+1,r}) S_m + \mathcal{O}(|S_{\lfloor N/Q_1 \rfloor}|)$$

with  $S_m = \sum_{n=1}^m B_{n,r}$ . We start by bounding  $A_{m,r}$ . Since

$$|A_{m,r} - A_{m+1,r}| = \mathcal{O}\left(\sum_{j=l+1}^d |\theta_j| Q_1 N^{j-1}\right),$$

by (3.8), we obtain

$$|A_{m,r} - A_{m+1,r}| = \mathcal{O}(N^{-1}(\log N)^{(d-l+1)\beta_1}).$$

To estimate  $S_m$  we are going to apply Lemma 3.1. Recall that  $\beta_1$  has been fixed in the minor case, we now set  $\beta' = l(d-l)\beta_1$  and  $Q = Q_1^l$ . Since  $Q_1 \leq (\log N)^{(d-l)\beta_1}$ , we have  $Q \leq (\log N)^{\beta'}$ . Let  $\alpha_2 = \alpha + (d-l+1)\beta_1$  and  $\beta_{\alpha_2}$  be determined by  $\alpha_2$  as in Step 1, and take

$$\beta_2 \geq \beta_{\alpha_2} = (\alpha_2 + 2)(2l^2 - 2l + 1).$$

Then, for  $\beta > 2\beta_2 + \beta'$  we have,  $\beta' < \beta$  and  $\beta_2 < \beta - \beta'$  and  $2\beta_2 < \beta$ , thus by Lemma 3.1, we obtain

$$\left| Q \xi_l - \frac{a'}{q'} \right| \leq \frac{1}{q'^2}$$

for some  $a'/q'$  such that  $(a', q') = 1$  and  $(\log N)^{\beta_2} \leq q' \leq N^j (\log N)^{-\beta_2}$ . Hence, by Step 1 applied to the polynomial  $x \mapsto \xi_l Q x^l + R(x)$ , we get

$$|S_m| = \mathcal{O}(m(\log m)^{-\alpha_2}).$$

Therefore, all together

$$\left| \sum_{n=1}^N e^{2\pi i P(n)} \varphi(n) \right| = \mathcal{O}(Q_1 N^{-1} (\log N)^{(d-l+1)\beta_1} N^2 Q_1^{-2} (\log N)^{-\alpha_2})$$

since

$$\sum_{m=1}^{\lfloor N/Q_1 \rfloor} m(\log m)^{-\alpha_2} = \mathcal{O}(N^2 Q_1^{-2} (\log N)^{-\alpha_2}).$$

Hence,

$$(3.10) \quad \left| \sum_{n=1}^N e^{2\pi i P(n)} \varphi(n) \right| = \mathcal{O}(N(\log N)^{(d-l+1)\beta_1 - \alpha_2}).$$

and (3.10) gives us the desired conclusion.

We now turn to the proof of Lemma 3.1

*Proof of Lemma 3.1.* We apply Dirichlet's theorem to  $Q\theta$ . Thus there exists  $a'/q'$ , such that  $(a', q') = 1$ , with

$$(3.11) \quad \left| Q\theta - \frac{a'}{q'} \right| \leq \frac{(\log N)^{\beta_2}}{q' N^j}$$

and  $q' \leq N^j (\log N)^{-\beta_2}$ . We must see that  $(\log N)^{\beta_2} \leq q'$ . There are two cases. First,  $a'/q' = Qa/q$ , then  $q' \geq q/Q \geq (\log N)^{\beta-\beta'} \geq (\log N)^{\beta_2}$ , and we are done. Second, we suppose  $a'/q' \neq Qa/q$ . Then

$$\frac{1}{qq'} \leq \left| \frac{a'}{q'} - \frac{Qa}{q} \right| \leq \left| Q\theta - \frac{a'}{q'} \right| + Q \left| \theta - \frac{a}{q} \right|.$$

But  $|Q\theta - a'/q'| \leq N^{-j} (\log N)^{\beta_2}$ , and  $|\theta - a/q| \leq q^{-2}$ , so by (3.11)

$$\frac{1}{qq'} \leq N^{-j} (\log N)^{\beta_2} + Qq^{-2}.$$

That is

$$\frac{1}{q'} \leq q N^{-j} (\log N)^{\beta_2} + Qq^{-1} \leq (\log N)^{\beta_2-\beta} + (\log N)^{\beta'-\beta},$$

since,  $(\log N)^\beta \leq q \leq N^j (\log N)^{-\beta}$ , and  $Q \leq (\log N)^{\beta'}$ . Thus

$$\frac{1}{q'} \leq (\log N)^{-\beta_2}.$$

This proves the lemma.  $\square$

**Step 3.** We assume  $k > 1$ ,

$$P(x) = \sum_{|\gamma| \leq d} \xi_\gamma x^\gamma,$$

and for some  $\gamma_0$  such that  $|\gamma_0| = d$  we have

$$\left| \xi_{\gamma_0} - \frac{a}{q} \right| \leq \frac{1}{q^2},$$

for some  $a/q$  with  $(a, q) = 1$  and  $(\log N)^\beta \leq q \leq N^d (\log N)^{-\beta}$ . We need the following lemma.

**Lemma 3.2** ([16, Lemma 1]). *For each  $\gamma_0$  such that  $|\gamma_0| = d$ , there exist  $\nu$  linear transformations  $L_1, \dots, L_\nu$  of  $\mathbb{R}^k$  that have integer coefficients and determinant 1 (so that each  $L_j$  is an automorphism of  $\mathbb{Z}^k$ ) and integers  $c_0, \dots, c_\nu$ , with  $c_0 \neq 0$ , so that if  $\theta$  is the coefficient of  $x^{\gamma_0}$  of  $P(x)$ , and  $\sigma_j$  is the coefficient of  $x_1^d$  of  $P(L_j x)$ , then*

$$c_0 \theta = c_1 \sigma_1 + \dots + c_\nu \sigma_\nu.$$

*The operators  $L_1, \dots, L_\nu$ , and integers  $c_0, \dots, c_\nu$ , depend only on  $k, d$ , and  $\gamma_0$ . Moreover,  $\nu$  is the dimension of the vector space of polynomials in  $\mathbb{R}^k$  which are homogeneous of degree  $d$ .*

We shall apply Lemma 3.2 with  $\theta = \xi_{\gamma_0}$ . Now, for  $\beta_1$  sufficiently large, determined below, apply Dirichlet's principle to each  $\sigma_j$  to get  $a_j/q_j$  so that  $(a_j, q_j) = 1$  and

$$(3.12) \quad \left| \sigma_j - \frac{a_j}{q_j} \right| \leq \frac{(\log N)^{\beta_1}}{q_j N^d}$$

and  $1 \leq q_j \leq N^d (\log N)^{-\beta_1}$ , for  $j = 1, \dots, \nu$ . There are two cases. The first case, the minor case, that is when  $q_j \geq (\log N)^{\beta_1}$  for at least one  $j \in \{1, \dots, \nu\}$ . In that case we write  $L = L_j$  and  $\tilde{P}(x) = P(L(x))$ ,  $\tilde{\varphi}(x) = \varphi(L(x))$  and  $\tilde{\Omega}_N = L^{-1}[\Omega_N]$ . We observe

$$(3.13) \quad \sum_{n \in \Omega_N \cap \mathbb{Z}^k} e^{2\pi i P(n)} \varphi(n) = \sum_{n \in \tilde{\Omega}_N \cap \mathbb{Z}^k} e^{2\pi i \tilde{P}(n)} \tilde{\varphi}(n)$$

and since  $\Omega_N \subseteq \{x \in \mathbb{R}^k : |x| \leq cN\}$  we have  $\tilde{\Omega}_N \subseteq \{x \in \mathbb{R}^k : |x| \leq \tilde{c}N\}$  for some  $\tilde{c} > 0$ .

Next, for each  $n \in \mathbb{Z}^k$ , write  $n = (n_1, n')$  where  $n_1 \in \mathbb{Z}$  and  $n' \in \mathbb{Z}^{k-1}$ . Then the sum

$$(3.14) \quad \sum_{n \in \tilde{\Omega}_N \cap \mathbb{Z}^k} e^{2\pi i \tilde{P}(n)} \tilde{\varphi}(n) = \sum_{\substack{n' \in \mathbb{Z}^{k-1} \\ |n'| \leq \tilde{c}N}} \sum_{\substack{n_1 \in \mathbb{Z} \\ (n_1, n') \in \tilde{\Omega}_N}} e^{2\pi i \tilde{P}(n_1, n')} \tilde{\varphi}(n_1, n').$$

Thus it suffices to have

$$(3.15) \quad \left| \sum_{\substack{n_1 \in \mathbb{Z} \\ (n_1, n') \in \tilde{\Omega}_N}} e^{2\pi i \tilde{P}(n_1, n')} \tilde{\varphi}(n_1, n') \right| = \mathcal{O}(N(\log N)^{-\alpha})$$

with the  $\mathcal{O}$  term independent of  $n'$ . This will give the desired result for (3.13), since there are at most  $\mathcal{O}(N^{k-1})$  terms that appear in the first summation in (3.14) as  $|n'| \leq \tilde{c}N$ . Now, apply Step 1 to the one-dimensional polynomial  $n_1 \mapsto \tilde{P}(n_1, n')$ . Observe that  $\tilde{P}(n_1, n') = \sigma_j n_1^d + \text{lower order terms of } n_1$ . We choose any  $\beta_1 \geq \beta_\alpha = (\alpha + 2)(2d^2 - 2d + 1)$  as in Step 1 and we obtain (3.15) which leads to the desired result.

The second case, the major case, where  $1 \leq q_j \leq (\log N)^{\beta_1}$  for all  $1 \leq j \leq \nu$ , will be seen to be empty, where  $\beta_1$  has been fixed and  $\beta$  is chosen large enough.

In fact, by Lemma 3.2

$$(3.16) \quad \theta = c_0^{-1}(c_1\sigma_1 + \dots + c_\nu\sigma_\nu),$$

so if we write  $a'/q' = c_0^{-1}(c_1a_1/q_1 + \dots + c_\nu a_\nu/q_\nu)$ , with  $(a', q') = 1$ , then  $q' \leq c(\log N)^{\beta'}$ , with  $\beta' = \nu\beta_1$  since  $q_j \leq (\log N)^{\beta_1}$  for all  $1 \leq j \leq \nu$ .

Now there are two subcases:  $a'/q' = a/q$  and  $a'/q' \neq a/q$ . The first is not possible, except for finitely many  $N$ , since

$$(\log N)^\beta \leq q = q' \leq c(\log N)^{\beta'}$$

and we may take  $\beta > \beta'$ . In the second, we write

$$\frac{1}{qq'} \leq \left| \frac{a'}{q'} - \frac{a}{q} \right| \leq \left| \theta - \frac{a}{q} \right| + \left| \theta - \frac{a'}{q'} \right| \leq \frac{1}{q^2} + CN^{-d}(\log N)^{\beta_1}$$

in account of (3.16) and (3.12). Thus

$$c^{-1}(\log N)^{-\beta'} \leq \frac{1}{q'} \leq \frac{1}{q} + CqN^{-d}(\log N)^{\beta_1} \leq C'((\log N)^{-\beta} + (\log N)^{\beta_1 - \beta})$$

since  $(\log N)^\beta \leq q \leq N^d(\log N)^{-\beta}$ . Thus, it is enough to take  $\beta > (\nu + 1)\beta_1$  to make this impossible, since  $\beta > \beta' = \nu\beta_1$  and  $\beta > \beta' + \beta_1$ . This concludes Step 3.

**Step 4.** We assume that  $k > 1$  and  $|\gamma_0| < d$ . This combines ideas of both Step 2 and Step 3. The argument is by backward induction on the degree of  $|\gamma_0|$ . The result from Step 3 establishes the first step of the backward induction for  $|\gamma_0| = d$ . Assume that Theorem 3.1 holds for all  $|\gamma_0| = j$  such that  $l < j \leq d$ . Our aim now is to deduce it holds for  $|\gamma_0| = l$ .

Write  $P = P_0 + P_1$ , with

$$P_0(x) = \sum_{l < |\gamma| \leq d} \xi_\gamma x^\gamma \quad \text{and} \quad P_1(x) = \sum_{|\gamma| \leq l} \xi_\gamma x^\gamma.$$

On  $P_1$  we apply Lemma 3.2 and argue as in Step 3. This gives us an automorphism  $L$  of  $\mathbb{Z}^k$ , so that if  $\tilde{P}(x) = P(Lx)$ ,  $\tilde{P}_0(x) = P_0(Lx)$ ,  $\tilde{P}_1(x) = P_1(Lx)$ , then  $\tilde{P}_1(x) = \theta x_1^l + R(x)$ , where  $R$  is a polynomial of degree  $\leq l$ , but of degree  $\leq l - 1$  in  $x_1$ . Also

$$\left| \theta - \frac{a_1}{q_1} \right| \leq \frac{(\log N)^{\beta_0}}{q_1 N^l}$$

with  $(a_1, q_1) = 1$ ,  $(\log N)^{\beta_0} \leq q_1 \leq N^l(\log N)^{-\beta_0}$  as long as

$$(3.17) \quad (\nu_1 + 1)\beta_0 \leq \beta$$

where  $\nu_1$  is the dimension of the space of homogeneous polynomials of degree  $l$  in  $\mathbb{R}^k$ . Let us emphasize that the parameter  $\beta_0$  in (3.17) plays the role of  $\beta_1$  from the previous step.

Next, we choose  $\beta_1 > 0$  whose value will be determined later, and apply Dirichlet's principle to all the coefficients  $\tilde{\xi}_\gamma$  of  $\tilde{P}_0$  with  $l < |\gamma| \leq d$ . Thus we can find  $a_\gamma/q_\gamma$ , so that

$$\left| \tilde{\xi}_\gamma - \frac{a_\gamma}{q_\gamma} \right| \leq \frac{(\log N)^{\beta_1}}{q_\gamma N^{|\gamma|}}$$

with  $(a_\gamma, q_\gamma) = 1$  and  $1 \leq q_\gamma \leq N^{|\gamma|}(\log N)^{-\beta_1}$ . We also set  $\theta_\gamma = \tilde{\xi}_\gamma - a_\gamma/q_\gamma$ . If  $(\log N)^{\beta_1} \leq q_\gamma$ , for some  $\gamma$  such that  $l < |\gamma| \leq d$ , then by the induction hypothesis, we obtain (3.3) provided that  $\beta_1$  is sufficiently large, and we are done. We fix this  $\beta_1$ .

Now let us define  $\beta' = \beta_1 \nu l$  where  $\nu = |\{\gamma \in \mathbb{N}_0^k : l < |\gamma| \leq d\}|$ , and  $\alpha_2 = (k+1)(\nu+1)\beta_1 + \alpha$  and  $\beta_{\alpha_2}$  be determined by  $\alpha_2$  as in Step 1, and take

$$(3.18) \quad \beta_2 \geq \beta_{\alpha_2} = (\alpha_2 + 2)(2l^2 - 2l + 1).$$

Suppose now that  $1 \leq q_\gamma \leq (\log N)^{\beta_1}$ , for all  $\gamma$  such that  $l < |\gamma| \leq d$ . Let  $q' = \text{lcm}(q_\gamma : l < |\gamma| \leq d)$ . Then  $q' \leq (\log N)^{\nu\beta_1}$ . Set  $Q = (q')^l$ , then  $Q \leq (\log N)^{\beta'}$ . Next, we choose

$$(3.19) \quad \beta_0 > 2\beta_2 + \beta'$$

then  $\beta_0 > \beta'$  and  $\beta_2 < \beta_0 - \beta'$  and  $2\beta_2 < \beta_0$  thus we can we apply Lemma 3.1 to get  $a'/q'$  so that

$$\left| Q\theta - \frac{a'}{q'} \right| \leq \frac{(\log N)^{\beta_2}}{q' N^l} \leq \frac{1}{q'^2}$$

with  $(a', q') = 1$ ,  $(\log N)^{\beta_2} \leq q' \leq N^l (\log N)^{-\beta_2}$ . We break the summation

$$\sum_{n \in \tilde{\Omega}_N \cap \mathbb{Z}^k} e^{2\pi i \tilde{P}(n)} \tilde{\varphi}(n)$$

into essentially a sum of disjoint boxes of side-length  $q'$  as follows. We will write  $n = (n_1, n')$  with  $n_1 \in \mathbb{Z}$ ,  $n' \in \mathbb{Z}^{k-1}$ . Set

$$\tilde{\Omega}_{N,q'} = \{m \in \mathbb{Z}^k : q'm + \mathbb{Z}_{q'}^k \subseteq \tilde{\Omega}_N\}.$$

So if we write  $n = q'm + r$ , then

$$\sum_{n \in \tilde{\Omega}_N \cap \mathbb{Z}^k} e^{2\pi i \tilde{P}(n)} \tilde{\varphi}(n) = \sum_{r \in \mathbb{Z}_{q'}^k} \sum_{m \in \tilde{\Omega}_{N,q'}} e^{2\pi i \tilde{P}(q'm+r)} \tilde{\varphi}(q'm+r) + \sum_{n \in \Delta} e^{2\pi i \tilde{P}(n)} \tilde{\varphi}(n).$$

The residual set of points  $\Delta$  are lattice points in  $\tilde{\Omega}_N$  whose distance from the boundary of  $\tilde{\Omega}_N$  is  $\mathcal{O}(q') = \mathcal{O}(N^{1-3\sigma})$  for any  $0 < \sigma < 1/3$ . Hence, by Proposition 3.1 there are  $\mathcal{O}(N^{k-\sigma})$  such points. Thus that sum contributes  $\mathcal{O}(N^{k-\sigma})$  which is  $\mathcal{O}(N^k (\log N)^{-\alpha})$ , for every  $\alpha > 0$ .

So we are reduced to considering

$$(3.20) \quad \sum_{r \in \mathbb{Z}_{q'}^k} \sum_{m \in \tilde{\Omega}_{N,q'}} e^{2\pi i \tilde{P}(q'm+r)} \tilde{\varphi}(q'm+r).$$

Let us fix  $r \in \mathbb{Z}_{q'}^k$  and  $m' \in \mathbb{Z}^{k-1}$ , and consider first one-dimensional sum

$$\sum_{\substack{m_1 \in \mathbb{Z} \\ (m_1, m') \in \tilde{\Omega}_{N,q'}}} e^{2\pi i \tilde{P}(q'm+r)} \tilde{\varphi}(q'm+r) = \sum_{m_1=M_0}^{M_1} A_{m_1} B_{m_1},$$

where  $\{M_0, \dots, M_1\} = \{m_1 \in \mathbb{Z} : (m_1, m') \in \tilde{\Omega}_{N,q'}\}$  and

$$A_{m_1} = e^{2\pi i \sum_{l < |\gamma| \leq d} \theta_\gamma (q'm+r)^\gamma}, \quad \text{and} \quad B_{m_1} = e^{2\pi i Q \theta m_1^l + R(m_1)} \tilde{\varphi}(q'm+r),$$

and  $R$  is a polynomial in  $m_1$  of degree  $\leq l-1$ , depending on  $r$  and  $m'$ . Now, by summation by parts we get

$$\sum_{m_1=M_0}^{M_1} A_{m_1} B_{m_1} = \sum_{m_1=M_0}^{M_1} (A_{m_1} - A_{m_1+1}) S_{m_1} + \mathcal{O}(|S_{M_1}|) + \mathcal{O}(|S_{M_0-1}|)$$

with  $S_m = \sum_{n=1}^m B_n$ . But

$$\begin{aligned} |A_{m_1} - A_{m_1+1}| &= \mathcal{O}\left(\sum_{l < |\gamma| \leq d} q' |\theta_\gamma| N^{|\gamma|-1}\right) \\ &= \mathcal{O}((\log N)^{\nu\beta_1} N^{|\gamma|-1} N^{-|\gamma|} (\log N)^{\beta_1}) \\ &= \mathcal{O}(N^{-1} (\log N)^{(\nu+1)\beta_1}). \end{aligned}$$

However,

$$S_{m_1} = \left| \sum_{n=1}^{m_1} B_n \right| = \mathcal{O}(m_1 (\log m_1)^{-\alpha_2}),$$

by the one-dimensional result applied to  $Q\theta$ , with  $\alpha_2 = (k+1)(\nu+1)\beta_1 + \alpha$ . By summation by parts

$$\left| \sum_{m_1=M_0}^{M_1} A_{m_1} B_{m_1} \right| = \mathcal{O}(N(\log N)^{(\nu+1)\beta_1 - \alpha_2}).$$

Thus when we also sum in  $m'$  and  $r$  we get that our sum (3.20) is

$$\mathcal{O}(N^k(\log N)^{(k+1)(\nu+1)\beta_1 - \alpha_2}) = \mathcal{O}(N^k(\log N)^{-\alpha}).$$

So by taking into account (3.17), (3.18) and (3.19), we need only to take  $\beta$  large enough, to make  $\beta_0$  large enough, to then make  $\beta_2$  large enough to get our desired conclusion.  $\square$

#### 4. $\ell^p$ -ESTIMATES

Let  $\mathcal{F}$  denote the Fourier transform on  $\mathbb{R}^d$  defined for any function  $f \in L^1(\mathbb{R}^d)$  as

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x) e^{2\pi i \xi \cdot x} dx.$$

If  $f \in \ell^1(\mathbb{Z}^d)$  we set

$$\hat{f}(\xi) = \sum_{x \in \mathbb{Z}^d} f(x) e^{2\pi i \xi \cdot x}.$$

To simplify the notation we denote by  $\mathcal{F}^{-1}$  the inverse Fourier transform on  $\mathbb{R}^d$  or the inverse Fourier transform on the torus  $\mathbb{T}^d \equiv [0, 1)^d$  (Fourier coefficients), depending on the context. Let  $\eta : \mathbb{R}^d \rightarrow \mathbb{R}$  be a smooth function such that  $0 \leq \eta(x) \leq 1$  and

$$\eta(x) = \begin{cases} 1 & \text{for } |x| \leq 1/(16d), \\ 0 & \text{for } |x| \geq 1/(8d). \end{cases}$$

*Remark 4.1.* We may additionally assume that  $\eta$  is a convolution of two non-negative smooth functions  $\phi$  and  $\psi$  with compact supports contained inside  $(-1/(8d), 1/(8d))^d$ .

**4.1. Useful tools.** Let  $(\Theta_N : N \in \mathbb{N})$  be a sequence of multipliers on  $\mathbb{R}^d$  with a property that for each  $p \in (1, \infty)$  there is a constant  $\mathbf{B}_p > 0$  such that for any  $(f_t : t \in \mathcal{Z}) \in L^p(\ell_{\mathcal{Z}}^2(\mathbb{R}^d)) \cap L^2(\ell_{\mathcal{Z}}^2(\mathbb{R}^d))$

$$(4.1) \quad \left\| \left( \sum_{t \in \mathcal{Z}} \mathcal{N}(\mathcal{F}^{-1}(\Theta_N \mathcal{F}f) : N \in \mathbb{N})^2 \right)^{1/2} \right\|_{L^p} \leq \mathbf{B}_p \left\| \left( \sum_{t \in \mathcal{Z}} |f_t|^2 \right)^{1/2} \right\|_{L^p},$$

where  $\mathcal{N}$  is a seminorm defined for sequences of complex numbers as in Section 2. Our aim is to prove from this its discrete analogue. See also the discussion of sampling in [11, Proposition 2.1 and Corollary 2.5]. Throughout this section we will assume that  $\mathcal{R}$  is a diagonal  $d \times d$  matrix with positive entries  $(r_\gamma : \gamma \in \Gamma)$  such that  $\inf_{\gamma \in \Gamma} r_\gamma \geq h$  for some  $h > 0$ . The next two lemmas are multi-dimensional analogues of Lemma 1 and Lemma 2 from [13].

**Lemma 4.1.** *For any  $h > \frac{1}{2}$  and  $u \in \mathbb{R}^d$*

$$(4.2) \quad \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} \eta(\mathcal{R}\xi) d\xi \right\|_{\ell^1(x)} \leq 1,$$

$$(4.3) \quad \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{2\pi i \xi \cdot u}) \eta(\mathcal{R}\xi) d\xi \right\|_{\ell^1(x)} \leq |\mathcal{R}^{-1}u|.$$

*The same result remains true if the Euclidean norm  $|\cdot|$  is replaced by  $|\cdot|_\infty$ .*

*Proof.* Recall that  $|x|_\infty \leq |x| \leq d|x|_\infty$  for any  $x \in \mathbb{R}^d$ . We only show the inequality (4.3) since the proof of (4.2) is almost identical. Let us observe, in view of Remark 4.1, that

$$\eta(\mathcal{R}\xi) = \det(\mathcal{R}) \phi_{\mathcal{R}} * \psi_{\mathcal{R}}(\xi)$$

where  $\phi_{\mathcal{R}}(\xi) = \phi(\mathcal{R}\xi)$ , and  $\psi_{\mathcal{R}}(\xi) = \psi(\mathcal{R}\xi)$ . For  $x \in \mathbb{Z}^d$  we have

$$\det(\mathcal{R})^{-1} \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{2\pi i \xi \cdot u}) \eta(\mathcal{R}\xi) d\xi = \mathcal{F}^{-1} \phi_{\mathcal{R}}(x) \mathcal{F}^{-1} \psi_{\mathcal{R}}(x) - \mathcal{F}^{-1} \phi_{\mathcal{R}}(x - u) \mathcal{F}^{-1} \psi_{\mathcal{R}}(x - u).$$



By Cauchy–Schwarz inequality and Plancherel’s theorem

$$\begin{aligned} \sum_{x \in \mathbb{Z}^d} |\mathcal{F}^{-1}\phi_{\mathcal{R}}(x)| \cdot |\mathcal{F}^{-1}\psi_{\mathcal{R}}(x) - \mathcal{F}^{-1}\psi_{\mathcal{R}}(x-u)| &\leq \|\mathcal{F}^{-1}\phi_{\mathcal{R}}\|_{\ell^2} \left\| \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} (1 - e^{2\pi i \xi \cdot u}) \psi_{\mathcal{R}}(\xi) d\xi \right\|_{\ell^2(x)} \\ &= \|\phi_{\mathcal{R}}\|_{L^2} \left\| (1 - e^{2\pi i \xi \cdot u}) \psi_{\mathcal{R}}(\xi) \right\|_{L^2(d\xi)}. \end{aligned}$$

Moreover, since  $|\cdot| \leq d|\cdot|_{\infty}$

$$\begin{aligned} \int_{\mathbb{R}^d} |1 - e^{-2\pi i \xi \cdot u}|^2 |\psi_{\mathcal{R}}(\xi)|^2 d\xi &\leq (2\pi d)^2 |\mathcal{R}^{-1}u|_{\infty}^2 \int_{\mathbb{R}^d} |\mathcal{R}\xi|^2 |\psi_{\mathcal{R}}(\xi)|^2 d\xi \\ &= (2\pi d)^2 \det(\mathcal{R})^{-1} |\mathcal{R}^{-1}u|_{\infty}^2 \int_{\mathbb{R}^d} |\xi|^2 |\psi(\xi)|^2 d\xi \end{aligned}$$

and the support of  $\psi$  is contained inside  $(-1/(8d), 1/(8d))^d$  we obtain

$$\sum_{x \in \mathbb{Z}^d} |\mathcal{F}^{-1}\phi_{\mathcal{R}}(x)| |\mathcal{F}^{-1}\psi_{\mathcal{R}}(x) - \mathcal{F}^{-1}\psi_{\mathcal{R}}(x-u)| \leq \det(\mathcal{R})^{-1} |\mathcal{R}^{-1}u|_{\infty} \|\phi\|_{L^2} \|\psi\|_{L^2}$$

which finishes the proof of (4.3) since  $\|\phi\|_{L^2} \|\psi\|_{L^2} \leq 2^{-1-d}$ .  $\square$

Let  $(X, \mu)$  be a measure space and suppose we are given  $p \geq 1$  and a family  $(T_m : m \in \mathbb{N}_0)$  of bounded linear operators  $T_m : L^p(X) \mapsto L^p(X)$ . Moreover, for each  $\omega \in [0, 1]$  we define

$$T^{\omega} = \sum_{m \in \mathbb{N}_0} \varepsilon_m(\omega) T_m$$

where  $(\varepsilon_m(\omega) : m \in \mathbb{N}_0)$  is the sequence of Rademacher functions on  $[0, 1]$ .

In connection with this we have the following vector-valued analogue of Marcinkiewicz–Zygmund inequality.

**Lemma 4.2.** *Suppose that for every  $p \in (1, \infty)$  there is a constant  $\mathbf{C}_p > 0$  such that for all  $\omega \in [0, 1]$  and  $f \in L^p(X)$*

$$(4.4) \quad \|T^{\omega} f\|_{L^p} \leq \mathbf{C}_p \|f\|_{L^p}.$$

*Then there is a constant  $C > 0$  such that for every sequence  $(f_t : t \in \mathcal{Z}) \in L^p(\ell_{\mathcal{Z}}^2(X))$*

$$(4.5) \quad \left\| \left( \sum_{t \in \mathcal{Z}} \sum_{m \in \mathbb{N}_0} |T_m f_t|^2 \right)^{1/2} \right\|_{L^p} \leq C \mathbf{C}_p \left\| \left( \sum_{t \in \mathcal{Z}} |f_t|^2 \right)^{1/2} \right\|_{L^p}.$$

*In particular, if  $T_m \equiv 0$  for each  $m \in \mathbb{N}$  then (4.5) implies the Marcinkiewicz–Zygmund result*

$$(4.6) \quad \left\| \left( \sum_{t \in \mathcal{Z}} |T_0 f_t|^2 \right)^{1/2} \right\|_{L^p} \leq C \mathbf{C}_p \left\| \left( \sum_{t \in \mathcal{Z}} |f_t|^2 \right)^{1/2} \right\|_{L^p}.$$

*Summation over  $\mathbb{N}_0$  in the inner sum of (4.5) can be replaced by any other countable set and the result remains valid.*

*Proof.* Let

$$F_{\omega'} = \sum_{t \in \mathcal{Z}} \varepsilon_t(\omega') f_t$$

where  $(\varepsilon_t(\omega') : t \in \mathcal{Z})$  is the sequence of Rademacher functions on  $[0, 1]$ . Then by the double Khinchine’s inequality we obtain

$$\left\| \left( \sum_{t \in \mathcal{Z}} \sum_{m \in \mathbb{N}_0} |T_m f_t|^2 \right)^{1/2} \right\|_{L^p}^p \lesssim \int_0^1 \int_0^1 \|T^{\omega}(F_{\omega'})\|_{L^p}^p d\omega d\omega'.$$

However,

$$\|T^{\omega}(F_{\omega'})\|_{L^p} \lesssim \|F_{\omega'}\|_{L^p}$$

for each  $\omega' \in [0, 1]$  by (4.4). Now

$$\int_0^1 \|F_{\omega'}\|_{L^p}^p d\omega' \lesssim \left\| \left( \sum_{t \in \mathcal{Z}} |f_t|^2 \right)^{1/2} \right\|_{L^p}^p$$

by Khinchine’s inequality and this proves the lemma.  $\square$

From now on, unless otherwise stated, we assume that every function  $f_t : \mathbb{Z}^d \rightarrow \mathbb{C}$  is finitely supported.

**Proposition 4.1.** *Under assumption (4.1) for each  $p \in (1, \infty)$  there is a constant  $C > 0$  such that for every  $h \geq 1$  we have*

$$\left\| \left( \sum_{t \in \mathbb{Z}} \mathcal{N}(\mathcal{F}^{-1}(\Theta_N \eta(\mathcal{R} \cdot) \hat{f}_t) : N \in \mathbb{N})^2 \right)^{1/2} \right\|_{\ell^p} \leq C \mathbf{B}_p \left\| \left( \sum_{t \in \mathbb{Z}} |\mathcal{F}^{-1}(\eta(\mathcal{R} \cdot) \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}.$$

*Proof.* Let  $\eta_{\mathcal{R}}(\xi) = \eta(\mathcal{R}\xi)$ . Since  $\eta_{\mathcal{R}} = \eta_{\mathcal{R}} \eta_{\mathcal{R}/2}$ , by Hölder's inequality we have

$$\begin{aligned} & \left( \sum_{t \in \mathbb{Z}} \mathcal{N}(\mathcal{F}^{-1}(\Theta_N \eta_{\mathcal{R}} \hat{f}_t)(x) : N \in \mathbb{N})^2 \right)^{p/2} \\ & \leq \left( \int_{\mathbb{R}^d} \left( \sum_{t \in \mathbb{Z}} \mathcal{N}(\mathcal{F}^{-1}(\Theta_N \eta_{\mathcal{R}} \hat{f}_t)(u) : N \in \mathbb{N})^2 \right)^{1/2} |\mathcal{F}^{-1} \eta_{\mathcal{R}/2}(x - u)| \, du \right)^p \\ & \leq \int_{\mathbb{R}^d} \left( \sum_{t \in \mathbb{Z}} \mathcal{N}(\mathcal{F}^{-1}(\Theta_N \eta_{\mathcal{R}} \hat{f}_t)(u) : N \in \mathbb{N})^2 \right)^{p/2} |\mathcal{F}^{-1} \eta_{\mathcal{R}/2}(x - u)| \, du \cdot \|\mathcal{F}^{-1} \eta_{\mathcal{R}/2}\|_{L^1}^{p-1}. \end{aligned}$$

Next, we note that  $\|\mathcal{F}^{-1} \eta_{\mathcal{R}/2}\|_{L^1} \lesssim 1$  and

$$\sum_{x \in \mathbb{Z}^d} |\mathcal{F}^{-1} \eta_{\mathcal{R}/2}(x - u)| \lesssim \det(\mathcal{R})^{-1} \sum_{x \in \mathbb{Z}^d} \frac{1}{(1 + |\mathcal{R}^{-1}(x - u)|^2)^{d+1}} \lesssim 1$$

which is uniformly bounded with respect to  $\mathcal{R}$ . Thus we obtain

$$\begin{aligned} \left\| \left( \sum_{t \in \mathbb{Z}} \mathcal{N}(\mathcal{F}^{-1}(\Theta_N \eta_{\mathcal{R}} \hat{f}_t) : N \in \mathbb{N})^2 \right)^{1/2} \right\|_{\ell^p} & \leq C \left\| \left( \sum_{t \in \mathbb{Z}} \mathcal{N}(\mathcal{F}^{-1}(\Theta_N \eta_{\mathcal{R}} \hat{f}_t) : N \in \mathbb{N})^2 \right)^{1/2} \right\|_{L^p} \\ & \leq C \mathbf{B}_p \left\| \left( \sum_{t \in \mathbb{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{L^p} \end{aligned}$$

where the last inequality is a consequence of (4.1). The proof will be completed if we show

$$\left\| \left( \sum_{t \in \mathbb{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{t \in \mathbb{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}.$$

For this purpose, using (4.3) from Lemma 4.1, we observe that

$$\begin{aligned} & \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{-2\pi i \xi \cdot u}) \eta_{\mathcal{R}/2}(\xi) \eta_{\mathcal{R}}(\xi) \hat{f}(\xi) \, d\xi \right\|_{\ell^p(x)} \\ & \leq \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{-2\pi i \xi \cdot u}) \eta_{\mathcal{R}/2}(\xi) \, d\xi \right\|_{\ell^1(x)}^p \|\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f})\|_{\ell^p} \lesssim \|\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f})\|_{\ell^p}. \end{aligned}$$

Therefore, in view of (4.6), we obtain

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^d} \int_{[0,1]^d} \left( \sum_{t \in \mathbb{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)(x + u) - \mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)(x)|^2 \right)^{p/2} \, du \\ & = \int_{[0,1]^d} \left\| \left( \sum_{t \in \mathbb{Z}} \left| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{-2\pi i \xi \cdot u}) \eta_{\mathcal{R}/2}(\xi) \eta_{\mathcal{R}}(\xi) \hat{f}_t(\xi) \, d\xi \right|^2 \right)^{1/2} \right\|_{\ell^p(x)}^p \\ & \leq \int_{[0,1]^d} \left\| \left( \sum_{t \in \mathbb{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}^p \, du \\ & \lesssim \left\| \left( \sum_{t \in \mathbb{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}^p. \end{aligned}$$

Hence,

$$\begin{aligned} \left\| \left( \sum_{t \in \mathbb{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{L^p}^p & = \sum_{x \in \mathbb{Z}^d} \int_{[0,1]^d} \left( \sum_{t \in \mathbb{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)(x + u)|^2 \right)^{p/2} \, du \\ & \lesssim \left\| \left( \sum_{t \in \mathbb{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}^p. \end{aligned}$$

This finishes the proof.  $\square$

**Lemma 4.3.** *Let  $p \in [1, \infty)$ , then for each  $Q \in \mathbb{N}$  and  $h \geq 2Q^{d+1}$  and any  $m \in \mathbb{N}_Q^d$  we have*

$$(4.7) \quad \|\mathcal{F}^{-1}(\eta(\mathcal{R} \cdot) \hat{f})(Qx + m)\|_{\ell^p(x)} \simeq Q^{-d/p} \|\mathcal{F}^{-1}(\eta(\mathcal{R} \cdot) \hat{f})\|_{\ell^p}.$$

Moreover, the implicit constants in (4.7) are independent of  $Q$ ,  $h$  and  $m$ .

*Proof.* Let  $\eta_{\mathcal{R}}(\xi) = \eta(\mathcal{R}\xi)$  as before. For each  $m \in \mathbb{N}_Q^d$  we set

$$J_m = \|\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f})(Qx + m)\|_{\ell^p(x)},$$

and  $I = \|\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f})\|_{\ell^p}$ . Then

$$\sum_{m \in \mathbb{N}_Q^d} J_m^p = I^p.$$

Since  $\eta_{\mathcal{R}} = \eta_{\mathcal{R}} \eta_{\mathcal{R}/2}$ , by Minkowski's inequality we obtain

$$\begin{aligned} & \|\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f})(Qx + m) - \mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f})(Qx + m')\|_{\ell^p(x)} \\ &= \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot (Qx + m)} (1 - e^{2\pi i \xi \cdot (m - m')}) \eta_{\mathcal{R}}(\xi) \hat{f}(\xi) d\xi \right\|_{\ell^p(x)} \\ &\leq \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{2\pi i \xi \cdot (m - m')}) \eta_{\mathcal{R}/2}(\xi) d\xi \right\|_{\ell^1(x)} I \\ &\leq |\mathcal{R}^{-1}(m - m')|_{\infty} I \end{aligned}$$

where in the last step we have used Lemma 4.1. Hence, for all  $m, m' \in \mathbb{N}_Q^d$

$$J_{m'} \leq J_m + Qh^{-1}I.$$

Thus

$$(4.8) \quad J_{m'}^p \leq 2^{p-1} J_m^p + 2^{p-1} Q^p h^{-p} I^p.$$

Therefore,

$$I^p = \sum_{m' \in \mathbb{N}_Q^d} J_{m'}^p \leq 2^{p-1} Q^d J_m^p + 2^{p-1} Q^{d+p} h^{-p} I^p.$$

By the assumptions

$$2^p Q^{d+p} h^{-p} = (2Q^{d/p+1} h^{-1})^p \leq 1.$$

Hence, we obtain  $I^p \leq 2^p Q^d J_m^p$ . For the converse inequality, we use again (4.8) to get

$$Q^d J_{m'}^p \leq 2^{p-1} \sum_{m \in \mathbb{N}_Q^d} J_m^p + 2^{p-1} Q^{d+p} h^{-p} I^p \leq 2^p I^p$$

and the proof of Lemma 4.3 is completed.  $\square$

**Proposition 4.2.** *Under assumption (4.1) for each  $p \in (1, \infty)$  there is a constant  $C > 0$  such that for each  $Q \in \mathbb{N}$  and  $h \geq 2Q^{d+1}$  and any  $m \in \mathbb{N}_Q^d$  we have*

$$\begin{aligned} & \left\| \left( \sum_{t \in \mathcal{Z}} \mathcal{N}(\mathcal{F}^{-1}(\Theta_N \eta(\mathcal{R} \cdot) \hat{f}_t)(Qx + m) : N \in \mathbb{N})^2 \right)^{1/2} \right\|_{\ell^p(x)} \\ & \leq C \mathbf{B}_p \left\| \left( \sum_{t \in \mathcal{Z}} |\mathcal{F}^{-1}(\eta(\mathcal{R} \cdot) \hat{f}_t)(Qx + m)|^2 \right)^{1/2} \right\|_{\ell^p(x)}. \end{aligned}$$

*Proof.* Let  $\eta_{\mathcal{R}}(\xi) = \eta(\mathcal{R}\xi)$ . For each  $m \in \mathbb{N}_Q^d$  we define

$$J_m = \left\| \left( \sum_{t \in \mathcal{Z}} \mathcal{N}(\mathcal{F}^{-1}(\Theta_N \eta_{\mathcal{R}} \hat{f}_t)(Qx + m) : N \in \mathbb{N})^2 \right)^{1/2} \right\|_{\ell^p(x)}.$$

Then, by Proposition 4.1,

$$I^p = \sum_{m \in \mathbb{N}_Q^d} J_m^p = \left\| \left( \sum_{t \in \mathcal{Z}} \mathcal{N}(\mathcal{F}^{-1}(\Theta_N \eta_{\mathcal{R}} \hat{f}_t) : N \in \mathbb{N})^2 \right)^{1/2} \right\|_{\ell^p}^p \lesssim \mathbf{B}_p^p \left\| \left( \sum_{t \in \mathcal{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}^p.$$

If  $m, m' \in \mathbb{N}_Q^d$  then we may write

$$\begin{aligned} & \left\| \left( \sum_{t \in \mathcal{Z}} \mathcal{N} \left( \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot (Qx+m)} (1 - e^{2\pi i \xi \cdot (m-m')}) \Theta_N(\xi) \eta_{\mathcal{R}}(\xi) \hat{f}_t(\xi) \, d\xi : N \in \mathbb{N} \right)^2 \right)^{1/2} \right\|_{\ell^p(x)} \\ & \leq \mathbf{B}_p \left\| \left( \sum_{t \in \mathcal{Z}} \left| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{2\pi i \xi \cdot (m-m')}) \eta_{\mathcal{R}}(\xi) \hat{f}_t(\xi) \, d\xi \right|^2 \right)^{1/2} \right\|_{\ell^p(x)}. \end{aligned}$$

Since  $\eta_{\mathcal{R}} = \eta_{\mathcal{R}} \eta_{\mathcal{R}/2}$ , by Lemma 4.1 and (4.6) the last expression may be dominated by

$$\begin{aligned} & \left\| \int_{\mathbb{T}^d} e^{-2\pi i \xi \cdot x} (1 - e^{2\pi i \xi \cdot (m-m')}) \eta_{\mathcal{R}/2}(\xi) \, d\xi \right\|_{\ell^1(x)} \left\| \left( \sum_{t \in \mathcal{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \\ & \leq Qh^{-1} \left\| \left( \sum_{t \in \mathcal{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}, \end{aligned}$$

thus

$$J_m \leq J_{m'} + \mathbf{B}_p Qh^{-1} \left\| \left( \sum_{t \in \mathcal{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}.$$

Raising to the  $p$ 'th power and summing up over  $m' \in \mathbb{N}_Q^d$  we get

$$Q^d J_m^p \leq 2^{p-1} I^p + 2^{p-1} \mathbf{B}_p^p Q^{d+p} h^{-p} \left\| \left( \sum_{t \in \mathcal{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}^p \lesssim \mathbf{B}_p^p \left\| \left( \sum_{t \in \mathcal{Z}} |\mathcal{F}^{-1}(\eta_{\mathcal{R}} \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}^p$$

and Lemma 4.3 finishes the proof.  $\square$

## 5. ESTIMATES FOR THE IONESCU–WAIINGER TYPE MULTIPLIERS

This section is intended to prove Theorem 5.1, which is inspired by the ideas of Ionescu and Wainger from [10]. Before we precisely formulate Theorem 5.1 we need some portion on notation. Let  $\rho > 0$  and for every  $N \in \mathbb{N}$  define

$$N_0 = \lfloor N^{\rho/2} \rfloor + 1 \quad \text{and} \quad Q_0 = (N_0!)^D$$

where  $D = D_\rho = \lfloor 2/\rho \rfloor + 1$ . Let  $\mathbb{P}$  denote the set of all prime numbers and  $\mathbb{P}_N = \mathbb{P} \cap (N_0, N]$ . For any  $V \subseteq \mathbb{P}_N$  we define

$$\Pi(V) = \bigcup_{k \in \mathbb{N}_D} \Pi_k(V)$$

where for any  $k \in \mathbb{N}_D$

$$\Pi_k(V) = \{p_1^{\gamma_1} \cdots p_k^{\gamma_k} : \gamma_l \in \mathbb{N}_D \text{ and } p_l \in V \text{ are distinct for all } 1 \leq l \leq k\}.$$

Therefore,  $\Pi_{k_1}(V) \cap \Pi_{k_2}(V) = \emptyset$  if  $k_1 \neq k_2$  and  $\Pi(V)$  is the set of all products of primes factors from  $V$  of length at most  $D$ , at powers between 1 and  $D$ . Now we introduce the sets

$$P_N = \{q = Q \cdot w : Q|Q_0 \text{ and } w \in \Pi(\mathbb{P}_N) \cup \{1\}\}.$$

It is not difficult to see that every integer  $q \in \mathbb{N}_N$  can be uniquely written as  $q = Q \cdot w$  where  $Q|Q_0$  and  $w \in \Pi(\mathbb{P}_N) \cup \{1\}$ . Moreover, if  $N \geq C_\rho$  for some  $C_\rho > 0$  then we obtain

$$q = Q \cdot w \leq Q_0 \cdot w \leq (N_0!)^D N^{D^2} \leq e^{N^\rho}$$

thus we have  $\mathbb{N}_N \subseteq P_N \subseteq \mathbb{N}_{e^{N^\rho}}$ . Furthermore, if  $N_1 \leq N_2$  then  $P_{N_1} \subseteq P_{N_2}$ .

For a subset  $S \subseteq \mathbb{N}$  we define

$$\mathcal{R}(S) = \{a/q \in \mathbb{T}^d \cap \mathbb{Q}^d : a \in A_q \text{ and } q \in S\}$$

where for each  $q \in \mathbb{N}$

$$A_q = \{a \in \mathbb{N}_q^d : \gcd(q, (a_\gamma : \gamma \in \Gamma)) = 1\}.$$

Finally, for each  $N \in \mathbb{N}$  we will consider

$$(5.1) \quad \mathcal{U}_N = \mathcal{R}(P_N).$$

It is easy to see, if  $N_1 \leq N_2$  then  $\mathcal{U}_{N_1} \subseteq \mathcal{U}_{N_2}$ .

We will assume that  $\Theta$  is a multiplier on  $\mathbb{R}^d$  and for every  $p \in (1, \infty)$  there is a constant  $\mathbf{A}_p > 0$  such that for every  $f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$  we have

$$(5.2) \quad \|\mathcal{F}^{-1}(\Theta \mathcal{F} f)\|_{L^p} \leq \mathbf{A}_p \|f\|_{L^p}.$$

For each  $N \in \mathbb{N}$  we define new periodic multipliers

$$\Delta_N(\xi) = \sum_{a/q \in \mathcal{U}_N} \Theta(\xi - a/q) \eta_N(\xi - a/q)$$

where  $\eta_N(\xi) = \eta(\mathcal{E}_N^{-1}\xi)$  and  $\mathcal{E}_N$  is a diagonal  $d \times d$  matrix with positive entries  $(\varepsilon_\gamma : \gamma \in \Gamma)$  such that  $\varepsilon_\gamma \leq e^{-N^{2\rho}}$ . The main result is the following.

**Theorem 5.1.** *Let  $\Theta$  be a multiplier on  $\mathbb{R}^d$  obeying (5.2). Then for every  $\rho > 0$  and  $p \in (1, \infty)$  there is a constant  $C_{\rho,p} > 0$  such that for any  $N \in \mathbb{N}$  and  $f \in \ell^p(\mathbb{Z}^d)$*

$$\|\mathcal{F}^{-1}(\Delta_N \hat{f})\|_{\ell^p} \leq C_{\rho,p} \mathbf{A}_p(\log N) \|f\|_{\ell^p}.$$

The main constructing blocks have been gathered in the next four subsections. Theorem 5.1 is a consequence of Theorem 5.3 and Proposition 5.1 proved below. The main idea of the proof of Theorem 5.1 is very simple, we find some  $C_\rho > 0$  and disjoint sets  $\mathcal{U}_N^i \subseteq \mathcal{U}_N$  such that

$$\mathcal{U}_N = \bigcup_{1 \leq i \leq C_\rho \log N} \mathcal{U}_N^i$$

and we show that  $\Delta_N$  with the summation restricted to  $\mathcal{U}_N^i$  is bounded on  $\ell^p(\mathbb{Z}^d)$  for every  $p \in (1, \infty)$ . This decomposition will be possible thanks to a suitable partition of integers from the set  $\Pi(\mathbb{P}_N) \cup \{1\}$ , see also [10]. Throughout this section we assume that  $f : \mathbb{Z}^d \rightarrow \mathbb{C}$  has a finite support.

**5.1. Fundamental combinatorial lemma.** Let us introduce the definition of sets with  $\mathcal{O}$  property.

**Definition 5.1.** A subset  $\Lambda \subseteq \Pi(V)$  has  $\mathcal{O}$  property if there is  $k \in \mathbb{N}_D$  and there are sets  $S_1, S_2, \dots, S_k$  with the following properties:

- (i) for each  $1 \leq j \leq k$  there is  $\beta_j \in \mathbb{N}$  such that  $S_j = \{q_{j,1}, \dots, q_{j,\beta_j}\}$ ;
- (ii) for every  $q_{j,s} \in S_j$  there are  $p_{j,s} \in V$  and  $\gamma_j \in \mathbb{N}_D$  such that  $q_{j,s} = p_{j,s}^{\gamma_j}$ ;
- (iii) for every  $w \in W$  there are unique numbers  $q_{1,s_1} \in S_1, \dots, q_{k,s_k} \in S_k$  such that  $w = q_{1,s_1} \dots q_{k,s_k}$ ;
- (iv) if  $(j, s) \neq (j', s')$  then  $(q_{j,s}, q_{j',s'}) = 1$ .

Now three comments are in order.

- (i) Firstly, we will assume that the set  $\Lambda = \{1\}$  has  $\mathcal{O}$  property corresponding to  $k = 0$ .
- (ii) Secondly, if  $\Lambda$  has  $\mathcal{O}$  property, then each subset  $\Lambda' \subseteq \Lambda$  has  $\mathcal{O}$  property as well.
- (iii) Thirdly, the most important remark is that, if a set  $\Lambda$  has  $\mathcal{O}$  property then each element of  $\Lambda$  has the same number of prime factors  $k \leq D$ .

The main result is the following.

**Lemma 5.1.** *For every  $\rho > 0$  there exists a constant  $C_\rho > 0$  such that for every  $N \in \mathbb{N}$  the set  $\mathcal{U}_N$  can be written as a disjoint union of at most  $C_\rho \log N$  sets  $\mathcal{U}_N^i = \mathcal{R}(P_N^i)$  where*

$$P_N^i = \{q = Q \cdot w : Q|Q_0 \text{ and } w \in \Lambda_i(\mathbb{P}_N)\}$$

and  $\Lambda_i(\mathbb{P}_N) \subseteq \Pi(\mathbb{P}_N) \cup \{1\}$  has  $\mathcal{O}$  property for each integer  $1 \leq i \leq C_\rho \log N$ .

The proof of Lemma 5.1 will be based on the following.

**Lemma 5.2.** *Suppose  $N, k \in \mathbb{N}$  are given with  $N \geq k$ . Then there exists a constant  $C_k > 0$  independent of  $N$ , and a family  $\pi = \{\pi_i(\mathbb{N}_N) : 1 \leq i \leq C_k \log N\}$  of partitions of  $\mathbb{N}_N$  into  $k$  non-empty subsets with the following properties:*

- (i) for every  $1 \leq i \leq C_k \log N$  each  $\pi_i(\mathbb{N}_N) = \{N_1^i, \dots, N_k^i\}$  consists of pairwise disjoint subsets of  $\mathbb{N}_N$  and  $\mathbb{N}_N = N_1^i \cup \dots \cup N_k^i$ ;
- (ii) for every  $E \subseteq \mathbb{N}_N$  with at least  $k$  elements there exists  $\pi_i(\mathbb{N}_N) = \{N_1^i, \dots, N_k^i\} \in \pi$  such that  $E \cap N_j^i \neq \emptyset$  for every  $1 \leq j \leq k$ .

The same conclusion holds for each  $V \subseteq \mathbb{N}_N$  with cardinality at least  $k$ .

*Proof.* We know that every surjection  $f : \mathbb{N}_N \mapsto \mathbb{N}_k$  determines a partition  $\{f^{-1}[\{m\}] : 1 \leq m \leq k\}$  of  $\mathbb{N}_N$  into  $k$  non-empty subsets. Therefore, we shall work with surjective mappings  $f$  from the set  $\mathbb{N}_N$  onto the set  $\mathbb{N}_k$  rather than with partitions of  $\mathbb{N}_N$  into  $k$  non-empty subsets. Let  $1 \leq k \leq N$  and let  $S(N, k)$  denote the set of all surjections  $f : \mathbb{N}_N \mapsto \mathbb{N}_k$ . Using inclusion-exclusion principle we can give the explicit formula for  $|S(N, k)|$ , but it is not important here. Let  $E \subseteq \mathbb{N}_N$  with cardinality  $k$  and note that

$$|\{f \in S(N, k) : |f[E]| = k\}| = k! k^{N-k} \geq \frac{k!}{k^k} |S(N, k)|$$

since  $k^N \geq |S(N, k)|$ . The proof will be completed if we show that for  $r = \lceil \frac{k^{k+1}}{k!} \log(\frac{eN}{k}) \rceil + 1 \simeq C_k \log N$  we have

$$(5.3) \quad \{(f_1, \dots, f_r) \in S(N, k)^r : \forall E \subseteq \mathbb{N}_N \exists 1 \leq m \leq r \ |f_m[E]| = k\} \neq \emptyset.$$

This means that it is possible to find among at most  $C_k \log N$  surjections at least one  $f : \mathbb{N}_N \mapsto \mathbb{N}_k$  such that  $|f[E]| = k$ . Then the set  $\{f^{-1}[\{m\}] : 1 \leq m \leq k\}$  is a partition of  $\mathbb{N}_N$  and  $E \cap f^{-1}[\{m\}] \neq \emptyset$  for every  $1 \leq m \leq k$  as desired.

Suppose for a contradiction that the set in (5.3) is empty. Then

$$\begin{aligned} |S(N, k)|^r &= |\{(f_1, \dots, f_r) \in S(N, k)^r : \exists E \subseteq \mathbb{N}_N \forall 1 \leq m \leq r \ |f_m[E]| < k\}| \\ &\leq \sum_{E \subseteq \mathbb{N}_N : |E|=k} |\{(f_1, \dots, f_r) \in S(N, k)^r : \forall 1 \leq m \leq r \ |f_m[E]| < k\}| \\ &= \sum_{E \subseteq \mathbb{N}_N : |E|=k} |\{f \in S(N, k) : |f[E]| < k\}|^r \\ &\leq \sum_{E \subseteq \mathbb{N}_N : |E|=k} \left(1 - \frac{k!}{k^k}\right)^r |S(N, k)|^r \\ &= \binom{N}{k} \left(1 - \frac{k!}{k^k}\right)^r |S(N, k)|^r \\ &\leq \left(\frac{eN}{k}\right)^k e^{-r \frac{k!}{k^k}} |S(N, k)|^r \\ &= e^{k \log(\frac{eN}{k}) - r \frac{k!}{k^k}} |S(N, k)|^r \end{aligned}$$

and this is impossible since  $e^{k \log(\frac{eN}{k}) - r \frac{k!}{k^k}} < 1$ , due to the fact  $r = \lceil \frac{k^{k+1}}{k!} \log(\frac{eN}{k}) \rceil + 1 > \frac{k^{k+1}}{k!} \log(\frac{eN}{k})$ .  $\square$

*Proof of Lemma 5.1.* We have to prove that for every  $V \subseteq \mathbb{P}_N$  the set  $\Pi(V)$  can be written as a disjoint union of at most  $C_k \log N$  sets with  $\mathcal{O}$  property. Fix  $k \in \mathbb{N}_D$ , let  $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{N}_D^k$  be a multi-index and observe that

$$\Pi_k(V) = \bigcup_{\gamma \in \mathbb{N}_D^k} \Pi_k^\gamma(V)$$

where

$$\Pi_k^\gamma(V) = \{p_1^{\gamma_1} \cdots p_k^{\gamma_k} : p_l \in V \text{ are distinct for all } 1 \leq l \leq k\}.$$

Since there are  $D^k$  possible choices of exponents  $\gamma_1, \dots, \gamma_k \in \mathbb{N}_D$  when  $k \in \mathbb{N}_D$ , it only suffices to prove that every  $\Pi_k^\gamma(V)$  can be partitioned into a union (not necessarily disjoint) of at most  $C_k \log N$  sets with  $\mathcal{O}$  property.

According to Lemma 5.2 for each integer  $1 \leq k \leq |V|$  there is a constant  $C_k > 0$  and a family  $\pi = \{\pi_i(V) : 1 \leq i \leq C_k \log |V|\}$  with the following properties:

- (i) for every  $1 \leq i \leq C_k \log |V|$  each  $\pi_i(V) = \{V_1^i, \dots, V_k^i\}$  consists of pairwise disjoint subsets of  $V$  and  $V = V_1^i \cup \dots \cup V_k^i$ ;
- (ii) for every  $E \subseteq V$  with at least  $k$  elements there exists  $\pi_i(V) = \{V_1^i, \dots, V_k^i\} \in \pi$  such that  $E \cap V_j^i \neq \emptyset$  for every  $1 \leq j \leq k$ .

Then one sees that for a fixed  $\gamma \in \mathbb{N}_D^k$  we have

$$(5.4) \quad \Pi_k^\gamma(V) = \bigcup_{1 \leq i \leq C_k \log |V|} \Pi_{k,i}^\gamma(V)$$

where

$$\Pi_{k,i}^\gamma(V) = \{p_1^{\gamma_1} \cdots p_k^{\gamma_k} : p_j^{\gamma_j} \in (V \cap V_j^i)^{\gamma_j} \text{ and } V_j^i \in \pi_i(V) \text{ for each } 1 \leq j \leq k\}$$

and  $(V \cap V_j^i)^{\gamma_j} = \{p^{\gamma_j} : p \in V \cap V_j^i\}$ . Indeed, the sum on the right-hand side of (5.4) is contained in  $\Pi_k^\gamma(V)$  since each  $\Pi_{k,i}^\gamma(V)$  is. For the opposite inclusion take  $p_1^{\gamma_1} \cdots p_k^{\gamma_k} \in \Pi_k^\gamma(V)$  and let  $E = \{p_1, \dots, p_k\}$ , then property (ii) for the family  $\pi$  ensures that there is  $\pi_i(V) = \{V_1^i, \dots, V_k^i\} \in \pi$  such that  $E \cap V_j^i \neq \emptyset$  for every  $1 \leq j \leq k$ . Therefore,  $p_1^{\gamma_1} \cdots p_k^{\gamma_k} \in \Pi_{k,i}^\gamma(V)$ . Furthermore, we see that for each  $1 \leq i \leq C_k \log N$  the sets  $\Pi_{k,i}^\gamma(V)$  have  $\mathcal{O}$  property. This completes the proof of Lemma 5.1.  $\square$

**5.2. Further reductions.** For each  $1 \leq i \leq C_\rho \log N$  we define

$$\Delta_N^i(\xi) = \sum_{a/q \in \mathcal{U}_N^i} \Theta(\xi - a/q) \eta_N(\xi - a/q)$$

with  $\mathcal{U}_N^i$  as in Lemma 5.1, and we see that

$$\Delta_N = \sum_{1 \leq i \leq C_\rho \log N} \Delta_N^i.$$

Therefore, the proof of Theorem 5.1 will be completed if we show that for every  $p \in (1, \infty)$  and  $\rho > 0$ , there is a constant  $C > 0$  such that for any  $N \in \mathbb{N}$  and  $1 \leq i \leq C_\rho \log N$  we have

$$(5.5) \quad \|\mathcal{F}^{-1}(\Delta_N^i \hat{f})\|_{\ell^p} \leq C \mathbf{A}_p \|f\|_{\ell^p}$$

for every  $f \in \ell^p(\mathbb{Z}^d)$ . In fact we show a slightly stronger result with no extra effort, which will imply inequality from (5.5). For this purpose, let

$$(5.6) \quad \Lambda \subseteq \Pi(\mathbb{P}_N) \cup \{1\}$$

be a set with  $\mathcal{O}$  property, see Definition 5.1. Moreover, we define

$$\mathcal{U}_N^\Lambda = \mathcal{R}(\{q = Q \cdot w : Q|Q_0 \text{ and } w \in \Lambda\})$$

and  $\mathcal{W}_N = \mathcal{R}(\Lambda)$ . Finally we introduce

$$\Delta_N^\Lambda(\xi) = \sum_{a/q \in \mathcal{U}_N^\Lambda} \Theta(\xi - a/q) \eta_N(\xi - a/q).$$

Our aim is to show that for every  $p \in (1, \infty)$  and  $\rho > 0$ , there is a constant  $C > 0$  such that for any  $N \geq 8^{\max\{p, p'\}/\rho}$  and for any set  $\Lambda$  as in (5.6) and for every  $f \in \ell^p(\mathbb{Z}^d)$  we have

$$(5.7) \quad \|\mathcal{F}^{-1}(\Delta_N^\Lambda \hat{f})\|_{\ell^p} \leq C \mathbf{A}_p \|f\|_{\ell^p}.$$

For  $N \leq 8^{\max\{p, p'\}/\rho}$  the bound in (5.7) is obvious, since we allow the constant  $C > 0$  to depend on  $p$  and  $\rho$ . Moreover, by the duality and interpolation, it suffices to prove (5.7) for  $p = 2r$  where  $r \in \mathbb{N}$ . If  $\Lambda = \Lambda_i(P_N)$  as in Lemma 5.1 for some  $1 \leq i \leq C_\rho \log N$ , then we see that  $\mathcal{U}_N^\Lambda = \mathcal{U}_N^i$  and  $\Delta_N^\Lambda = \Delta_N^i$ , and consequently (5.7) implies (5.5) as desired.

For the proof of (5.7) we expect to use the strong orthogonality between the supports of the functions  $\eta_N(\xi - a/q)$  in the multiplier  $\Delta_N^\Lambda(\xi)$  when  $a/q$  varies over  $\mathcal{U}_N^\Lambda$  with the set  $\Lambda$  as in (5.6). We will achieve this by exploiting  $\mathcal{O}$  property for the set  $\Lambda$ . For do so, further reductions are necessary.

The function  $\Theta(\xi) \eta_N(\xi)$  is regarded as a periodic function on  $\mathbb{T}^d$ , thus

$$\begin{aligned} \Delta_N^\Lambda(\xi) &= \sum_{a/q \in \mathcal{U}_N^\Lambda} \Theta(\xi - a/q) \eta_N(\xi - a/q) \\ &= \sum_{b \in \mathbb{N}_{Q_0}^d} \sum_{a/w \in \mathcal{W}_N} \Theta(\xi - b/Q_0 - a/w) \eta_N(\xi - b/Q_0 - a/w). \end{aligned}$$

In the last line we have used the fact, that if  $(q_1, q_2) = 1$  then for every  $a \in \mathbb{Z}^d$ , there are unique  $a_1, a_2 \in \mathbb{Z}^d$ , such that  $a_1/q_1, a_2/q_2 \in [0, 1)^d$  and

$$(5.8) \quad \frac{a}{q_1 q_2} = \frac{a_1}{q_1} + \frac{a_2}{q_2} \pmod{\mathbb{Z}^d}.$$

Since  $\Lambda$  has  $\mathcal{O}$  property then according to Definition 5.1 there is an integer  $1 \leq k \leq 2/\rho + 1$  and there are sets  $S_1, \dots, S_k$  such that for any  $j \in \mathbb{N}_k$  we have  $S_j = \{q_{j,1}, \dots, q_{j,\beta_j}\}$  for some  $\beta_j \in \mathbb{N}$ .

Now for each  $j \in \mathbb{N}_k$  we introduce

$$\mathcal{U}_{\{j\}} = \left\{ \frac{a_{j,s}}{q_{j,s}} \in \mathbb{T}^d \cap \mathbb{Q}^d : s \in \mathbb{N}_{\beta_j} \text{ and } a_{j,s} \in A_{q_{j,s}} \right\}$$

and for any  $M = \{j_1, \dots, j_m\} \subseteq \mathbb{N}_k$  let

$$\mathcal{U}_M = \{u_{j_1} + \dots + u_{j_m} \in \mathbb{T}^d \cap \mathbb{Q}^d : u_{j_l} \in \mathcal{U}_{\{j_l\}} \text{ for any } l \in \mathbb{N}_m\}.$$

For any sequence  $\sigma = (s_{j_1}, \dots, s_{j_m}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}$  determined by the set  $M$ , let us define

$$\mathcal{V}_M^\sigma = \left\{ \frac{a_{j_1, s_{j_1}}}{q_{j_1, s_{j_1}}} + \dots + \frac{a_{j_m, s_{j_m}}}{q_{j_m, s_{j_m}}} \in \mathbb{T}^d \cap \mathbb{Q}^d : a_{j_l, s_{j_l}} \in A_{q_{j_l, s_{j_l}}} \text{ for any } l \in \mathbb{N}_m \right\}.$$



Note that  $\mathcal{V}_M^\sigma$  is a subset of  $\mathcal{U}_M$  with fixed denominators  $q_{j_1, s_{j_1}}, \dots, q_{j_m, s_{j_m}}$ . If  $M = \emptyset$  then we have  $\mathcal{U}_M = \mathcal{V}_M = \{0\}$ . Let

$$\chi(\xi) = \mathbb{1}_\Lambda(\xi) \quad \text{and} \quad \Omega_N(\xi) = \Theta(\xi)\eta_N(\xi).$$

Then again by (5.8) we obtain

$$\begin{aligned} (5.9) \quad \Delta_N^\Lambda(\xi) &= \sum_{a/w \in \mathcal{W}_N} \sum_{b \in \mathbb{N}_{Q_0}^d} \Theta(\xi - b/Q_0 - a/w) \eta_N(\xi - b/Q_0 - a/w) \\ &= \sum_{s_1 \in \mathbb{N}_{\beta_1}} \sum_{a_{1,s_1} \in A_{q_{1,s_1}}} \cdots \sum_{s_k \in \mathbb{N}_{\beta_k}} \sum_{a_{k,s_k} \in A_{q_{k,s_k}}} \chi(q_{1,s_1} \cdots q_{k,s_k}) \sum_{b \in \mathbb{N}_{Q_0}^d} \Omega_N\left(\xi - b/Q_0 - \sum_{j=1}^k a_{j,s_j}/q_{j,s_j}\right) \\ &= \sum_{s_1 \in \mathbb{N}_{\beta_1}} \sum_{a_{1,s_1} \in A_{q_{1,s_1}}} \cdots \sum_{s_k \in \mathbb{N}_{\beta_k}} \sum_{a_{k,s_k} \in A_{q_{k,s_k}}} m_{a_{1,s_1}/q_{1,s_1} + \dots + a_{k,s_k}/q_{k,s_k}}(\xi) = \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} m_u(\xi) \end{aligned}$$

where for  $u = a_{1,s_1}/q_{1,s_1} + \dots + a_{k,s_k}/q_{k,s_k}$

$$m_u(\xi) = m_{a_{1,s_1}/q_{1,s_1} + \dots + a_{k,s_k}/q_{k,s_k}}(\xi) = \chi(q_{1,s_1} \cdots q_{k,s_k}) \sum_{b \in \mathbb{N}_{Q_0}^d} \Omega_N\left(\xi - b/Q_0 - \sum_{j=1}^k a_{j,s_j}/q_{j,s_j}\right).$$

For the simplicity of notation we will write from now on, for every  $u \in \mathcal{U}_{\mathbb{N}_k}$ , that

$$(5.10) \quad f_u(x) = \mathcal{F}^{-1}(m_u \hat{f})(x)$$

with  $f \in \ell^{2r}(\mathbb{Z}^d)$  and  $r \in \mathbb{N}$ . Therefore,

$$\mathcal{F}^{-1}(\Delta_N^\Lambda \hat{f})(x) = \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u(x)$$

and the proof of inequality (5.7) will follow from the following.

**Theorem 5.2.** *Suppose that  $\rho > 0$  and  $r \in \mathbb{N}$  are given. Then there is a constant  $C_{\rho,r} > 0$  such that for any  $N > 8^{2r/\rho}$  and for any set  $\Lambda$  as in (5.6) and for every  $f \in \ell^{2r}(\mathbb{Z}^d)$  we have*

$$(5.11) \quad \left\| \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u \right\|_{\ell^{2r}} \leq C_{\rho,r} \mathbf{A}_{2r} \|f\|_{\ell^{2r}}.$$

Moreover, the integer  $k \in \mathbb{N}_D$ , the set  $\mathcal{U}_{\mathbb{N}_k}$  and consequently the sets  $S_1, \dots, S_k$  are determined by the set  $\Lambda$  as it was described above.

**5.3. Strong orthogonality and estimates of square functions.** We have reduced the matters to proving Theorem 5.2. The bound (5.11) will follow from Theorem 5.3 and Proposition 5.1 formulated below. Theorem 5.3, which is the main result of this subsection, shows that one can control the left-hand side of (5.11) by a sum of  $\ell^{2r}(\mathbb{Z}^d)$  norms of appropriate square functions corresponding to some subsums of  $\sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u$ , see (5.30). The idea of proof with square function estimates originates in Ionescu and Wainger's paper [10], see Lemma 2.1 and Lemma 2.2 given there. The proof of Theorem 5.3 consists of two steps, see Lemma 5.5 and Lemma 5.6 below, which applied recursively will result in (5.30).

We now introduce a suitable square function which will be useful in bounding (5.11). For any  $M \subseteq \mathbb{N}_k$  and  $L = \{j_1, \dots, j_l\} \subseteq M$  and any sequence  $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}}$  determined by the set  $L$  let us define the following square function  $\mathcal{S}_{L,M}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k})$  associated with the sequence  $(f_u : u \in \mathcal{U}_{\mathbb{N}_k})$  of complex-valued functions as in (5.10), by setting

$$(5.12) \quad \mathcal{S}_{L,M}^\sigma(f_u(x) : u \in \mathcal{U}_{\mathbb{N}_k}) = \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \sum_{u \in \mathcal{U}_{M \setminus L}} \sum_{v \in \mathcal{V}_L^\sigma} f_{w+u+v}(x) \right|^2 \right)^{1/2},$$

where  $M^c = \mathbb{N}_k \setminus M$ . We will use the convention that for some  $s_{j_i} \in \{s_{j_1}, \dots, s_{j_l}\}$

$$\left\| \mathcal{S}_{L,M}^\sigma(f_u(x) : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell_{s_{j_i}}^2} = \left( \sum_{s_{j_i} \in \mathbb{N}_{\beta_{j_i}}} \left| \mathcal{S}_{L,M}^{(s_{j_1}, \dots, s_{j_l})}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})(x) \right|^2 \right)^{1/2}$$

and defines some function which depends on  $x \in \mathbb{Z}^d$  and on each  $s_{j_n} \in \{s_{j_1}, \dots, s_{j_l}\} \setminus \{s_{j_i}\}$ .

As it was mentioned above, for (5.11) we have to exploit the fact that the Fourier transform of  $f_u$  is defined as a sum of disjointly supported smooth cut-off functions. This observation suggests that appropriate subsums of  $\sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u$  should be strongly orthogonal to each other. To extract these strong orthogonality properties efficiently we have to introduce, as in [10], the definition of sequences with the uniqueness property.

**Definition 5.2.** A finite sequence  $(x_1, x_2, \dots, x_m)$  has the uniqueness property if there is  $k \in \mathbb{N}_m$  such that  $x_l \neq x_k$  for any  $l \in \mathbb{N}_m \setminus \{k\}$ .

Now fix  $j_n \in M \setminus L$  and let  $\mathcal{A}_{\beta_{j_n}} = \bigcup_{t=1}^{\beta_{j_n}} A_{q_{j_n}, t}$ . For any  $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}}$  determined by the set  $L$  we define for  $w \in \mathcal{U}_{M^c}$

$$(5.13) \quad F_h^{\sigma, w}(x) = \sum_{u' \in \mathcal{U}_{M \setminus (L \cup \{j_n\})}} \sum_{v \in \mathcal{V}_L^\sigma} \sum_{a_h \in A_{q_{j_n}, h}} f_{w+u'+v+a_h/q_{j_n}, h}(x)$$

and

$$(5.14) \quad \mathcal{F}_{a/q_{j_n}, h}^{\sigma, w}(x) = \sum_{u' \in \mathcal{U}_{M \setminus (L \cup \{j_n\})}} \sum_{v \in \mathcal{V}_L^\sigma} \mathbb{1}_{A_{q_{j_n}, h}}(a) f_{w+u'+v+a/q_{j_n}, h}(x).$$

Note that  $F_h^{\sigma, w}$  and  $\mathcal{F}_{a/q_{j_n}, h}^{\sigma, w}$  are appropriate pieces in the definition of the square function from (5.12). Namely,

$$\sum_{u \in \mathcal{U}_{M \setminus L}} \sum_{v \in \mathcal{V}_L^\sigma} f_{w+u+v}(x) = \sum_{h \in \mathbb{N}_{\beta_{j_n}}} F_h^{\sigma, w}(x)$$

and

$$F_h^{\sigma, w}(x) = \sum_{a \in \mathcal{A}_{\beta_{j_n}}} \mathcal{F}_{a/q_{j_n}, h}^{\sigma, w}(x).$$

The next lemma with Definition 5.2 will explain their orthogonality properties.

**Lemma 5.3.** Under the assumptions of Theorem 5.2, suppose that  $M \subseteq \mathbb{N}_k$  and  $L = \{j_1, \dots, j_l\} \subseteq M$ . Fix  $j_n \in M \setminus L$  and let  $\mathcal{A}_{\beta_{j_n}} = \bigcup_{t=1}^{\beta_{j_n}} A_{q_{j_n}, t}$  and let  $F_h^{\sigma, w}$  and  $\mathcal{F}_{a/q_{j_n}, h}^{\sigma, w}$  be the functions defined in (5.13) and (5.14) respectively. If  $(h_1, h_2, \dots, h_{2r-1}, h_{2r}) \in \mathbb{N}_{\beta_{j_n}}^{2r}$  is a sequence with the uniqueness property then

$$(5.15) \quad \sum_{x \in \mathbb{Z}^d} F_{h_1}^{\sigma, w_1}(x) \overline{F_{h_2}^{\sigma, w_1}(x)} \dots F_{h_{2r-1}}^{\sigma, w_r}(x) \overline{F_{h_{2r}}^{\sigma, w_r}(x)} = 0$$

for every  $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}}$  determined by the set  $L$  and every  $(w_1, \dots, w_r) \in \mathcal{U}_{M^c}^r$ .

Additionally, if  $(a_1, a_2, \dots, a_{2r-1}, a_{2r}) \in \mathcal{A}_{\beta_{j_n}}^{2r}$  is a sequence with the uniqueness property then

$$(5.16) \quad \sum_{x \in \mathbb{Z}^d} \mathcal{F}_{a_1/q_{j_n}, h_1}^{\sigma, w_1}(x) \overline{\mathcal{F}_{a_2/q_{j_n}, h_1}^{\sigma, w_1}(x)} \dots \mathcal{F}_{a_{2r-1}/q_{j_n}, h_r}^{\sigma, w_r}(x) \overline{\mathcal{F}_{a_{2r}/q_{j_n}, h_r}^{\sigma, w_r}(x)} = 0$$

for every  $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}}$  determined by the set  $L$ , every  $(w_1, \dots, w_r) \in \mathcal{U}_{M^c}^r$  and  $1 \leq h_1 < \dots < h_r \leq \beta_{j_n}$ .

*Proof.* Fix  $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}}$  and  $(w_1, \dots, w_r) \in \mathcal{U}_{M^c}^r$ . We firstly prove (5.15). Let  $(h_1, h_2, \dots, h_{2r-1}, h_{2r}) \in \mathbb{N}_{\beta_{j_n}}^{2r}$  be the sequence with the uniqueness property. By Definition 5.2 we can assume, without of loss of generality, that  $h_1 \neq h_i$  for any  $i \in \mathbb{N}_{2r} \setminus \{1\}$ . Set  $\mathcal{S}_1 = \text{supp } \mathcal{F}^{-1}(F_{h_1}^{\sigma, w})$ . Let

$$\mathbf{Q}_{\rho, N} = \prod_{\gamma \in \Gamma} (- (8d)^{-1} \cdot \varepsilon_\gamma, (8d)^{-1} \cdot \varepsilon_\gamma)$$

be a cube and note that, generally, we have

$$(5.17) \quad \text{supp } \mathcal{F}^{-1}(F_h^{\sigma, w}) \subseteq \bigcup_{b \in \mathbb{N}_{Q_0}} \bigcup_{a_h \in A_{q_{j_n}, h}} \bigcup_{u' \in \mathcal{U}_{M \setminus (L \cup \{j_n\})}} \bigcup_{v \in \mathcal{V}_L^\sigma} \frac{b}{Q_0} + \frac{a_h}{q_{j_n}, h} + u' + v + w + \mathbf{Q}_{\rho, N}.$$

Moreover, each  $b/Q_0 + a_h/q_{j_n}, h + u' + v + w$  in the sum above defines a rational fraction in  $\mathbb{Q}^d$  with a denominator bounded by  $e^{N^\rho}$  from above.

Now we see that  $\mathcal{F}^{-1}(\overline{F_{h_2}^{\sigma,w_1}} \dots \overline{F_{h_{2r-1}}^{\sigma,w_r}} \overline{F_{h_{2r}}^{\sigma,w_r}})$  is supported in the algebraic sum of  $2r-1$  sets as on the right-hand side of (5.17) determined by  $h_2, \dots, h_{2r-1}, h_{2r} \in \mathbb{N}_{\beta_{j_n}}$ . Thus

$$\mathcal{S}_2 = \text{supp } \mathcal{F}^{-1}(\overline{F_{h_2}^{\sigma,w_1}} \dots \overline{F_{h_{2r-1}}^{\sigma,w_r}} \overline{F_{h_{2r}}^{\sigma,w_r}}) \subseteq \bigcup_{a/W} \frac{a}{W} + (2r-1)\mathbf{Q}_{\rho,N}$$

where  $a/W$  varies over some subset of  $\mathbb{Q}^d$  with the property that all denominators  $W$  are relatively prime with  $q_{j_n, h_1}$  and  $W \leq e^{(2r-1)N^\rho}$ .

We are going to prove that  $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ . Suppose for a contradiction that there is  $\xi \in \mathcal{S}_1 \cap \mathcal{S}_2$  and there are  $a_1/W_1$  and  $a_2/W_2$  associated with  $\mathcal{S}_1$  and  $\mathcal{S}_2$  respectively such that  $W_1 \leq e^{N^\rho}$ ,  $W_2 \leq e^{(2r-1)N^\rho}$  and

$$\left| \xi_\gamma - \frac{a_{1,\gamma}}{W_1} \right| \leq (8d)^{-1} \cdot e^{-N^{2\rho}}, \quad \text{and} \quad \left| \xi_\gamma - \frac{a_{2,\gamma}}{W_2} \right| \leq (8d)^{-1} \cdot (2r-1) \cdot e^{-N^{2\rho}}$$

since  $\varepsilon_\gamma \leq e^{-N^{2\rho}}$  for every  $\gamma \in \Gamma$ . Since  $(q_{j_n, h_1}, W_2) = 1$ , there is  $\gamma \in \Gamma$  such that  $a_{1,\gamma}/W_1 \neq a_{2,\gamma}/W_2$ . Indeed,  $a_1/W_1 = a_{h_1}/q_{j_n, h_1} + a'_1/W'_1$  for some  $W'_1 \in \mathbb{N}$  such that  $(q_{j_n, h_1}, W'_1) = 1$ . If  $a_1/W_1 = a_2/W_2$ , then

$$\frac{a_{h_1}}{q_{j_n, h_1}} = \frac{a_2}{W_2} - \frac{a'_1}{W'_1} = \frac{W'_1 a_2 - W_2 a'_1}{W_2 W'_1},$$

but this is impossible. Thus for some  $\gamma \in \Gamma$  we have

$$e^{-2rN^\rho} \leq \left| \frac{a_{1,\gamma}}{W_1} - \frac{a_{2,\gamma}}{W_2} \right| \leq r \cdot e^{-N^{2\rho}}$$

and consequently  $N^{2\rho} - 2rN^\rho \leq \log r$ , but we have assumed that  $8^r < N^\rho$ , hence

$$2^r \leq 2^r(2^{3r} - 2r) \leq N^\rho(N^\rho - 2r) \leq \log r$$

which gives contradiction. We have proven that  $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$  and Plancherel's theorem yields (5.15).

In order to prove (5.16) we will proceed in a similar way as above. Namely, assume that the sequence  $(a_1, a_2, \dots, a_{2r-1}, a_{2r}) \in \mathcal{A}_{\beta_{j_n}}^{2r}$  has the uniqueness property and assume that  $a_1 \neq a_i$  for any  $i \in \mathbb{N}_{2r} \setminus \{1\}$ . Let us define  $\mathcal{S}_1 = \text{supp } \mathcal{F}^{-1}(\mathcal{F}_{a_1/q_{j_n, h_1}}^{\sigma, w_1})$ . Generally, note that

$$(5.18) \quad \text{supp } \mathcal{F}^{-1}(\mathcal{F}_{a/q_{j_n, h}}^{\sigma, w}) \subseteq \bigcup_{b \in \mathbb{N}_{Q_0}} \bigcup_{u' \in \mathcal{U}_{M \setminus (L \cup \{j_n\})}} \bigcup_{v \in \mathcal{V}_L^\sigma} \frac{b}{Q_0} + u' + v + w + \frac{a}{q_{j_n, h}} + \mathbf{Q}_{\rho, N}.$$

Each  $b/Q_0 + u' + v + w + a/q_{j_n, h}$  in the sum above defines a rational fraction in  $\mathbb{Q}^d$  with a denominator bounded by  $e^{N^\rho}$  from above. Furthermore,  $\mathcal{F}^{-1}(\overline{\mathcal{F}_{a_2/q_{j_n, h_1}}^{\sigma, w_1}} \overline{\mathcal{F}_{a_3/q_{j_n, h_2}}^{\sigma, w_2}} \dots \overline{\mathcal{F}_{a_{2r-1}/q_{j_n, h_r}}^{\sigma, w_r}} \overline{\mathcal{F}_{a_{2r}/q_{j_n, h_r}}^{\sigma, w_r}})$  is supported in the algebraic sum of  $2r-1$  sets as on the right-hand side of (5.18) determined by  $h_1, h_2, \dots, h_r \in \mathbb{N}_{\beta_{j_n}}$ . Thus

$$\mathcal{S}_2 = \text{supp } \mathcal{F}^{-1}(\overline{\mathcal{F}_{a_2/q_{j_n, h_1}}^{\sigma, w_1}} \overline{\mathcal{F}_{a_3/q_{j_n, h_2}}^{\sigma, w_2}} \dots \overline{\mathcal{F}_{a_{2r-1}/q_{j_n, h_r}}^{\sigma, w_r}} \overline{\mathcal{F}_{a_{2r}/q_{j_n, h_r}}^{\sigma, w_r}}) \subseteq \bigcup_{B/W} \frac{B}{W} + (2r-1)\mathbf{Q}_{\rho, N}$$

where  $B/W$  varies over some subset of  $\mathbb{Q}^d$  with the property that  $W \leq e^{(2r-1)N^\rho}$ . Now we show that  $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ . If not then there is  $\xi \in \mathcal{S}_1 \cap \mathcal{S}_2$  and there are

$$\frac{B_1}{W_1} = \frac{a_1}{q_{j_n, h_1}} + \frac{B'_1}{W'_1}, \quad \text{and} \quad \frac{B_2}{W_2} = \frac{a_2}{q_{j_n, h_1}} + \frac{B'_2}{W'_2}$$

associated with  $\mathcal{S}_1$  and  $\mathcal{S}_2$  respectively such that  $a_1 \neq a_2$ ,  $(q_{j_n, h_1}, W'_1) = 1$ ,  $(q_{j_n, h_1}, W'_2) = 1$ ,  $W_1 \leq e^{N^\rho}$ ,  $W_2 \leq e^{(2r-1)N^\rho}$  and for every  $\gamma \in \Gamma$

$$\left| \xi_\gamma - \frac{B_{1,\gamma}}{W_1} \right| \leq (8d)^{-1} \cdot e^{-N^{2\rho}}, \quad \text{and} \quad \left| \xi_\gamma - \frac{B_{2,\gamma}}{W_2} \right| \leq (8d)^{-1} \cdot (2r-1) \cdot e^{-N^{2\rho}}.$$

Thus, since  $a_1 \neq a_2$ , we obtain that  $B_1/W_1 \neq B_2/W_2$ . Indeed, if  $B_1/W_1 = B_2/W_2$  then

$$0 \neq \frac{a_1 - a_2}{q_{j_n, h_1}} = \frac{W'_1 B'_2 - W'_2 B'_1}{W'_1 W'_2}$$

but it is impossible since  $(q_{j_n, h_1}, W'_1 W'_2) = 1$ . Therefore, there is  $\gamma \in \Gamma$

$$e^{-2rN^\rho} \leq \left| \frac{B_{1,\gamma}}{W_1} - \frac{B_{2,\gamma}}{W_2} \right| \leq r \cdot e^{-N^{2\rho}}$$

which implies that  $2^r \leq \log r$ . This contradiction shows that  $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$  and consequently by Plancherel's theorem (5.16) follows and the proof of Lemma 5.3 is completed.  $\square$

In Lemma 5.5 and Lemma 5.6 we shall establish two fundamental properties of the square function  $\mathcal{S}_{L,M}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k})$  which will be critical for the proof of Theorem 5.3. However, we begin with a simple combinatorial lemma.

**Lemma 5.4.** *Let  $r \in \mathbb{N}$  and  $Z$  be a subset of a finite set  $X$  such that  $|Z| \leq r$ . Suppose we are given a sequence of complex numbers  $(g_{i,j} : i \in X, j \in \mathbb{N}_r)$ . Then for every sequence  $I = (i_0(1), i_1(1), \dots, i_0(r), i_1(r))$  which does not have the uniqueness property and*

$$(5.19) \quad \{i_0(1), i_1(1), \dots, i_0(r), i_1(r)\} = Z$$

*we have the following inequality*

$$|g_{i_0(1),1} \bar{g}_{i_1(1),1} \dots g_{i_0(r),r} \bar{g}_{i_1(r),r}| \leq 2 \sum_{\{i(1), \dots, i(r)\} = Z} |g_{i(1),1}|^2 \dots |g_{i(r),r}|^2.$$

*Proof.* We begin with the following claim. For every set  $Z$  with at most  $r$  elements if the sequence  $I = (i_0(1), i_1(1), \dots, i_0(r), i_1(r))$  does not have the uniqueness property and satisfies (5.19) then there are two functions  $\beta, \gamma : \{1, \dots, r\} \rightarrow \{0, 1\}$  such that  $\gamma(j) = 1 - \beta(j)$  and

$$\{i_{\beta(1)}(1), \dots, i_{\beta(r)}(r)\} = \{i_{\gamma(1)}(1), \dots, i_{\gamma(r)}(r)\} = Z.$$

We describe some algorithm which will be essential to establish the claim. For this purpose let us consider a quadruple  $(B_\beta, B_\gamma, P, m)$ , which consists of two subsets  $B_\beta, B_\gamma$  of  $\mathbb{N}$ , a set

$$P = \{(u_0(1), u_1(1)), \dots, (u_0(r), u_1(r))\}$$

of pairs of positive integers and

$$m = \min\{u_\sigma(j) : \sigma \in \{0, 1\} \text{ and } (u_0(j), u_1(j)) \in P\}.$$

We begin with a construction of a loop  $\mathcal{L}$  which will map quadruples  $(B_\beta, B_\gamma, P, m)$  to new quadruples  $(B'_\beta, B'_\gamma, P', m')$  with the properties as above, i.e.  $\mathcal{L}(B_\beta, B_\gamma, P, m) = (B'_\beta, B'_\gamma, P', m')$ .

Indeed, fix a quadruple  $(B_\beta, B_\gamma, P, m)$  as above. Let  $l_1 \in \mathbb{N}_r$  be the minimal number for which  $u_\sigma(l_1) = m$  for some  $\sigma \in \{0, 1\}$ , then we define  $\beta(l_1) = \sigma$  and  $\gamma(l_1) = 1 - \sigma$ . Suppose that we have constructed  $l_1, \dots, l_k \in \mathbb{N}_r$  distinct numbers and the functions  $\beta(l_1), \dots, \beta(l_k)$  and  $\gamma(l_1), \dots, \gamma(l_k)$  with the properties that  $l_j \in \mathbb{N}_r \setminus \{l_1, \dots, l_{j-1}\}$  is the smallest number for which  $u_{\beta(l_j)}(l_j) = u_{\gamma(l_{j-1})}(l_{j-1})$  for each  $2 \leq j \leq k$ .

We stop the loop  $\mathcal{L}(B_\beta, B_\gamma, P, m)$

- a) when we reach the first  $k \geq 1$  such that  $u_{\gamma(l_k)}(l_k) = u_{\beta(l_1)}(l_1)$ , or
- b) when we reach  $k \geq 1$  such that  $u_{\gamma(l_k)}(l_k)$  does not appear as the element of any pair in the set

$$P' = P \setminus \{(u_{\beta(l_1)}(l_1), u_{\gamma(l_1)}(l_1)), \dots, (u_{\beta(l_k)}(l_k), u_{\gamma(l_k)}(l_k))\}.$$

Now we define the sets

$$B'_\beta = B_\beta \cup \{u_{\beta(l_1)}(l_1), \dots, u_{\beta(l_k)}(l_k)\} \quad \text{and} \quad B'_\gamma = B_\gamma \cup \{u_{\gamma(l_1)}(l_1), \dots, u_{\gamma(l_k)}(l_k)\}.$$

From the first case follows

$$\{u_{\beta(l_1)}(l_1), \dots, u_{\beta(l_k)}(l_k)\} = \{u_{\gamma(l_1)}(l_1), \dots, u_{\gamma(l_k)}(l_k)\}.$$

The second case yields that  $u_{\beta(l_1)}(l_1) \neq u_{\gamma(l_k)}(l_k)$  and

$$\{u_{\beta(l_2)}(l_2), \dots, u_{\beta(l_k)}(l_k)\} = \{u_{\gamma(l_1)}(l_1), \dots, u_{\gamma(l_{k-1})}(l_{k-1})\}.$$

Moreover, if  $u_{\gamma(l_k)}(l_k)$  appears in the sequence  $(u_0(1), u_1(1), \dots, u_0(r), u_1(r))$  at least twice, then it must have been selected in  $\{u_{\gamma(l_1)}(l_1), \dots, u_{\gamma(l_{k-1})}(l_{k-1})\}$ . Hence  $\mathcal{L}(B_\beta, B_\gamma, P, m) = (B'_\beta, B'_\gamma, P', m')$  where

$$m' = \min\{u_\sigma(j) : \sigma \in \{0, 1\} \text{ and } (u_0(j), u_1(j)) \in P'\}.$$

Now we can describe the algorithm which will consists of three steps.

**Step 1.** Let  $P = \{(i_0(1), i_1(1)), \dots, (i_0(r), i_1(r))\}$  be the set of pairs corresponding to the sequence  $I$ , and  $m = \min\{i_\sigma(l) : l \in \mathbb{N}_r, \sigma \in \{0, 1\}\} = 1$ . We apply the loop  $\mathcal{L}$  to the quadruple  $(A_\beta, A_\gamma, P, m)$  with  $A_\beta = A_\gamma = \emptyset$ , which will result in a new quadruple  $(A_{\beta_0}^{k_0}, A_{\gamma_0}^{k_0}, P_{k_0}, m_{k_0})$ , where

$$P_{k_0} = P \setminus \{(i_{\beta(l_1)}(l_1), i_{\gamma(l_1)}(l_1)), \dots, (i_{\beta(l_{k_0})}(l_{k_0}), i_{\gamma(l_{k_0})}(l_{k_0}))\},$$

$m_{k_0} = \min\{i_\sigma(j) : \sigma \in \{0, 1\} \text{ and } (i_0(j), i_1(j)) \in P_{k_0}\}$  and the sets

$$A_\beta^{k_0} = \{i_{\beta(l_1)}(l_1), \dots, i_{\beta(l_{k_0})}(l_{k_0})\} \quad \text{and} \quad A_\gamma^{k_0} = \{i_{\gamma(l_1)}(l_1), \dots, i_{\gamma(l_{k_0})}(l_{k_0})\}.$$

The first application of the loop  $\mathcal{L}$  to the quadruple  $(A_\beta, A_\gamma, P, m)$  always results in  $A_\beta^{k_0} = A_\gamma^{k_0}$ , (see the case a) above), since the set  $P$  corresponds to the sequence  $I$ , whose each element appears at least twice. Therefore, if  $k_0 = |Z|$  we obtain the desired partition of the sequence  $I$ .

**Step 2.** Otherwise, we have  $k_0 < |Z|$ , and then we apply the loop  $\mathcal{L}$  with the data  $(A_\beta^{k_0}, A_\gamma^{k_0}, P_{k_0}, m_{k_0})$ , then we obtain a new quadruple  $(A_\beta^{k_1}, A_\gamma^{k_1}, P_{k_1}, m_{k_1})$ , where

$$P_{k_1} = P \setminus \{(i_{\beta(l_1)}(l_1), i_{\gamma(l_1)}(l_1)), \dots, (i_{\beta(l_{k_1})}(l_{k_1}), i_{\gamma(l_{k_1})}(l_{k_1}))\}$$

and  $m_{k_1} = \min\{i_\sigma(j) : \sigma \in \{0, 1\} \text{ and } (i_0(j), i_1(j)) \in P_{k_1}\}$  and the sets

$$A_\beta^{k_1} = A_\beta^{k_0} \cup \{i_{\beta(l_{k_0+1})}(l_{k_0+1}), \dots, i_{\beta(l_{k_1})}(l_{k_1})\} \quad \text{and} \quad A_\gamma^{k_1} = A_\gamma^{k_0} \cup \{i_{\gamma(l_{k_0+1})}(l_{k_0+1}), \dots, i_{\gamma(l_{k_1})}(l_{k_1})\}.$$

Observe if  $i_{\gamma(l_{k_1})}(l_{k_1}) = i_{\beta(l_{k_0+1})}(l_{k_0+1})$  and  $k_1 = |Z|$  then we are done. If  $k_1 < |Z|$  then we repeat the algorithm described in the Step 2. with the data  $(A_\beta^{k_1}, A_\gamma^{k_1}, P_{k_1}, m_{k_1})$  instead of  $(A_\beta^{k_0}, A_\gamma^{k_0}, P_{k_0}, m_{k_0})$ .

Assume now that  $i_{\gamma(l_{k_1})}(l_{k_1}) \neq i_{\beta(l_{k_0+1})}(l_{k_0+1})$ , this corresponds to the case b) described above with  $k = k_1$ . If  $i_{\beta(l_{k_0+1})}(l_{k_0+1}) \in A_\beta^{k_0}$  then  $A_\beta^{k_1} = A_\gamma^{k_1}$ , since  $k_1 \geq 1$  is that  $i_{\gamma(l_{k_1})}(l_{k_1})$  does not appear as the element of any pair in the set  $P_{k_1}$ , therefore  $i_{\gamma(l_{k_1})}(l_{k_1}) \in A_\gamma^{k_1}$  as well as  $i_{\gamma(l_{k_1})}(l_{k_1}) \in A_\beta^{k_1}$ , since every element in the sequence  $I$  appears at least twice. In this situation if  $k_1 = |Z|$  we obtain the claim. Otherwise, we repeat the Step 2. with the data  $(A_\beta^{k_1}, A_\gamma^{k_1}, P_{k_1}, m_{k_1})$  instead of  $(A_\beta^{k_0}, A_\gamma^{k_0}, P_{k_0}, m_{k_0})$ .

**Step 3.** Suppose now that  $i_{\gamma(l_{k_1})}(l_{k_1}) \neq i_{\beta(l_{k_0+1})}(l_{k_0+1})$ , and  $i_{\beta(l_{k_0+1})}(l_{k_0+1}) \notin A_\beta^{k_0}$ , then it is easy to see that  $A_\beta^{k_1} \setminus \{i_{\beta(l_{k_0+1})}(l_{k_0+1})\} = A_\gamma^{k_1}$ , since arguing in a similar way as above we show that  $i_{\gamma(l_{k_1})}(l_{k_1}) \in A_\gamma^{k_1}$  and  $i_{\gamma(l_{k_1})}(l_{k_1}) \in A_\beta^{k_1} \setminus \{i_{\beta(l_{k_0+1})}(l_{k_0+1})\}$ , due to the construction of the loop  $\mathcal{L}$ , see the case b) above.

So far we have applied the loop  $\mathcal{L}$  to quadruples which have the first two sets being equal. Now we apply  $\mathcal{L}$  with  $A_\gamma^{k_1} \neq A_\beta^{k_1}$  to the quadruple  $(A_\gamma^{k_1}, A_\beta^{k_1}, P_{k_1}, m_{k_1})$  with interchanged order of the sets  $A_\beta^{k_1}, A_\gamma^{k_1}$ , i.e.  $A_\gamma^{k_1}$  must be the first and  $A_\beta^{k_1}$  the second. We obtain a new quadruple  $(A_\gamma^{k_2}, A_\beta^{k_2}, P_{k_2}, m_{k_2})$  with

$$P_{k_2} = P \setminus \{(i_{\beta(l_1)}(l_1), i_{\gamma(l_1)}(l_1)), \dots, (i_{\beta(l_{k_2})}(l_{k_2}), i_{\gamma(l_{k_2})}(l_{k_2}))\}$$

and  $m_{k_2} = \min\{i_\sigma(j) : \sigma \in \{0, 1\} \text{ and } (i_0(j), i_1(j)) \in P_{k_2}\}$  and the sets

$$A_\beta^{k_2} = A_\beta^{k_1} \cup \{i_{\beta(l_{k_1+1})}(l_{k_1+1}), \dots, i_{\beta(l_{k_2})}(l_{k_2})\} \quad \text{and} \quad A_\gamma^{k_2} = A_\gamma^{k_1} \cup \{i_{\gamma(l_{k_1+1})}(l_{k_1+1}), \dots, i_{\gamma(l_{k_2})}(l_{k_2})\}.$$

Observe that  $m_{k_2} = i_{\beta(l_{k_0+1})}(l_{k_0+1})$  since  $i_{\beta(l_{k_0+1})}(l_{k_0+1})$  has not been considered in  $A_\beta^{k_0}$  and every element in the sequence  $I$  appears at least twice. Moreover, by the construction we have  $i_{\gamma(l_{k_1+1})}(l_{k_1+1}) = i_{\beta(l_{k_0+1})}(l_{k_0+1})$  and we see that

$$A_\beta^{k_1} = A_\gamma^{k_1} \cup \{i_{\gamma(l_{k_1+1})}(l_{k_1+1})\}.$$

Hence, if  $i_{\gamma(l_{k_1+1})}(l_{k_1+1}) = i_{\beta(l_{k_2})}(l_{k_2})$  then we have  $A_\beta^{k_2} = A_\gamma^{k_2}$ . If  $i_{\gamma(l_{k_1+1})}(l_{k_1+1}) \neq i_{\beta(l_{k_2})}(l_{k_2})$  we are in the case b) with  $k = k_2$  in the construction of  $\mathcal{L}$ , thus  $i_{\beta(l_{k_2})}(l_{k_2})$  must have been chosen before the choice  $i_{\beta(l_{k_2})}(l_{k_2})$ , since every element in the sequence  $I$  appears at least two times. This proves that  $i_{\beta(l_{k_2})}(l_{k_2}) \in A_\beta^{k_2}$  and consequently  $A_\beta^{k_2} = A_\gamma^{k_2}$ . Therefore, if  $k_2 = |Z|$  we obtain the desired partition of the sequence  $I$ . Otherwise, we restart our algorithm in the Step 2. with the quadruple  $(A_\beta^{k_2}, A_\gamma^{k_2}, P_{k_2}, m_{k_2})$  instead of  $(A_\beta^{k_0}, A_\gamma^{k_0}, P_{k_0}, m_{k_0})$ .

The advantage of the algorithm described in Step 1.-Step 3. is that after each iteration we obtain a quadruple  $(A'_\gamma, A'_\beta, P', m')$  such that  $A'_\gamma = A'_\beta$  and the cardinality of  $P'$  is smaller than the cardinality of the set of pairs  $P$  associated with the sequence  $I$ .

Thus after at most  $|Z|$  iterations we obtain two sets  $A_\beta^k, A_\gamma^k$  so that  $A_\beta^k = A_\gamma^k$  and  $P_k = \emptyset$ . Furthermore, for every pair  $(i_0(j), i_1(j)) \in P$  we obtain  $i_\sigma(j) \in A_\beta^k$  and  $i_{1-\sigma}(j) \in A_\gamma^k$  for some  $\sigma \in \{0, 1\}$ . Then  $A_\beta^k = A_\gamma^k = Z$  with the functions  $\beta$  and  $\gamma$  as desired. Having constructed the functions  $\beta$  and  $\gamma$  we see

$$|g_{i_0(1), 1} \bar{g}_{i_1(1), 1} \dots g_{i_0(r), r} \bar{g}_{i_1(r), r}| = (|g_{i_{\beta(1)}(1), 1}| \dots |g_{i_{\beta(r)}(r), r}|) (|g_{i_{\gamma(1)}(1), 1}| \dots |g_{i_{\gamma(r)}(r), r}|).$$

Therefore,

$$\begin{aligned} |g_{i_0(1),1}\bar{g}_{i_1(1),1} \cdots g_{i_0(r),r}\bar{g}_{i_1(r),r}| &\leq |g_{i_{\beta(1)}(1),1}|^2 \cdots |g_{i_{\beta(r)}(r),r}|^2 + |g_{i_{\gamma(1)}(1),1}|^2 \cdots |g_{i_{\gamma(r)}(r),r}|^2 \\ &\leq 2 \sum_{\{i(1),\dots,i(r)\}=Z} |g_{i(1),1}|^2 \cdots |g_{i(r),r}|^2. \end{aligned}$$

The proof of Lemma 5.4 is completed.  $\square$

**Lemma 5.5.** *Under the assumptions of Theorem 5.2, there is a constant  $C_r > 0$  such that for any  $M \subseteq \mathbb{N}_k$  and  $L = \{j_1, \dots, j_l\} \subseteq M$  and  $j_n \in M \setminus L$  and for any  $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}}$  determined by the set  $L$  we have*

$$\|\mathcal{S}_{L,M}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}} \leq C_r \left\| \|\mathcal{S}_{L \cup \{j_n\},M}^{\sigma \oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell_{s_{j_n}}^2} \right\|_{\ell^{2r}}$$

where  $\sigma \oplus s_{j_n} = (s_{j_1}, \dots, s_{j_l}, s_{j_n}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}} \times \mathbb{N}_{\beta_{j_n}}$  is the sequence determined by the set  $L \cup \{s_{j_n}\}$ .

*Proof.* Fix  $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}}$  determined by the set  $L$  and  $j_n \in M \setminus L$ . The set  $U_{M \setminus (L \cup \{j_n\})}$  is regarded as a subset of  $\mathcal{U}_{M \setminus L}$ . Thus

$$\begin{aligned} (5.20) \quad \|\mathcal{S}_{L,M}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} &= \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \sum_{h \in \mathbb{N}_{\beta_{j_n}}} F_h^{\sigma,w}(x) \right|^2 \right)^r \\ &= \sum_{(w_1, \dots, w_r) \in \mathcal{U}_{M^c}^r} \sum_{x \in \mathbb{Z}^d} \left| \sum_{h \in \mathbb{N}_{\beta_{j_n}}} F_h^{\sigma,w_1}(x) \right|^2 \cdots \left| \sum_{h \in \mathbb{N}_{\beta_{j_n}}} F_h^{\sigma,w_r}(x) \right|^2 \\ &= \sum_{(w_1, \dots, w_r) \in \mathcal{U}_{M^c}^r} \sum_{(h_0(1), h_1(1), \dots, h_0(r), h_1(r)) \in \mathbb{N}_{\beta_{j_n}}^{2r}} \sum_{x \in \mathbb{Z}^d} F_{h_0(1)}^{\sigma,w_1}(x) \overline{F_{h_1(1)}^{\sigma,w_1}(x)} \cdots F_{h_0(r)}^{\sigma,w_r}(x) \overline{F_{h_1(r)}^{\sigma,w_r}(x)}. \end{aligned}$$

Let us introduce for any set  $Z \subseteq \mathbb{N}_{\beta_{j_n}}$  with  $|Z| \leq 2r$

$$(5.21) \quad S_Z(x) = \sum F_{h_0(1)}^{\sigma,w_1}(x) \overline{F_{h_1(1)}^{\sigma,w_1}(x)} \cdots F_{h_0(r)}^{\sigma,w_r}(x) \overline{F_{h_1(r)}^{\sigma,w_r}(x)}$$

where the summation is taken over all sequences  $(h_0(1), h_1(1), \dots, h_0(r), h_1(r)) \in \mathbb{N}_{\beta_{j_n}}^{2r}$  which do not have the uniqueness property and  $\{h_0(1), h_1(1), \dots, h_0(r), h_1(r)\} = Z$ .

If the sequence  $(h_0(1), h_1(1), \dots, h_0(r), h_1(r)) \in \mathbb{N}_{\beta_{j_n}}^{2r}$  has the uniqueness property then (5.15) holds. Therefore,

$$(5.22) \quad \sum_{(h_0(1), h_1(1), \dots, h_0(r), h_1(r)) \in \mathbb{N}_{\beta_{j_n}}^{2r}} \sum_{x \in \mathbb{Z}^d} \prod_{i=1}^r F_{h_0(i)}^{\sigma,w_i}(x) \overline{F_{h_1(i)}^{\sigma,w_i}(x)} = \sum_{|Z| \leq r} \sum_{x \in \mathbb{Z}^d} S_Z(x).$$

The summation in the last sum is taken over all  $Z \subseteq \mathbb{N}_{\beta_{j_n}}$  such that  $|Z| \leq r$ , since by the uniqueness property,  $\sum_{x \in \mathbb{Z}^d} S_Z(x) = 0$  unless  $|Z| \leq r$ . By Lemma 5.4 we see that for each term in the sum from (5.21) we have

$$\left| F_{h_0(1)}^{\sigma,w_1}(x) \overline{F_{h_1(1)}^{\sigma,w_1}(x)} \cdots F_{h_0(r)}^{\sigma,w_r}(x) \overline{F_{h_1(r)}^{\sigma,w_r}(x)} \right| \leq 2 \sum_{\{h(1), \dots, h(r)\}=Z} \prod_{i=1}^r |F_{h(i)}^{\sigma,w_i}(x)|^2.$$

Thus

$$\begin{aligned} (5.23) \quad \sum_{x \in \mathbb{Z}^d} \sum_{|Z| \leq r} S_Z(x) &\lesssim_r \sum_{x \in \mathbb{Z}^d} \sum_{|Z| \leq r} \sum_{\{h(1), \dots, h(r)\}=Z} \prod_{i=1}^r |F_{h(i)}^{\sigma,w_i}(x)|^2 \\ &= \sum_{x \in \mathbb{Z}^d} \left( \sum_{h \in \mathbb{N}_{\beta_{j_n}}} |F_h^{\sigma,w_1}(x)|^2 \right) \cdots \left( \sum_{h \in \mathbb{N}_{\beta_{j_n}}} |F_h^{\sigma,w_r}(x)|^2 \right) \end{aligned}$$

since for any  $Z \subseteq \mathbb{N}_{\beta_{j_n}}$  the sum in (5.21) contains at most  $C_r$  terms.

Therefore, combining (5.20), (5.22) and (5.23) we obtain

$$\begin{aligned} \|\mathcal{S}_{L,M}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} &\lesssim_r \sum_{x \in \mathbb{Z}^d} \left( \sum_{h \in \mathbb{N}_{\beta_{j_n}}} \sum_{w \in \mathcal{U}_{M^c}} |F_h^{\sigma,w}(x)|^2 \right)^r \\ &= \left\| \|\mathcal{S}_{L \cup \{j_n\},M}^{\sigma \oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell_{s_{j_n}}^2} \right\|_{\ell^{2r}}^{2r}. \end{aligned}$$

This completes the proof of Lemma 5.5.  $\square$

**Lemma 5.6.** *Under the assumptions of Theorem 5.2, there is a constant  $C_r > 0$  such that for any  $M \subseteq \mathbb{N}_k$  and  $L = \{j_1, \dots, j_l\} \subseteq M$  and  $j_n \in M \setminus L$  and for any  $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}}$  determined by the set  $L$  we have*

$$\begin{aligned} \left\| \left\| \mathcal{S}_{L \cup \{j_n\}, M}^{\sigma \oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell_{s_{j_n}}^2} \right\|_{\ell^{2r}}^{2r} &\leq C_r \sum_{s_{j_n} \in \mathbb{N}_{\beta_{j_n}}} \left\| \mathcal{S}_{L \cup \{j_n\}, M}^{\sigma \oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} \\ &\quad + C_r \left\| \mathcal{S}_{L, M \setminus \{j_n\}}^{\sigma}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} \end{aligned}$$

where  $\sigma \oplus s_{j_n} = (s_{j_1}, \dots, s_{j_l}, s_{j_n}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}} \times \mathbb{N}_{\beta_{j_n}}$  is the sequence determined by the set  $L \cup \{j_n\}$ .

*Proof.* Recall from [10], Lemma 2.3, that there is a constant  $C_r > 0$  such that for any  $n \in \mathbb{N}$  and arbitrary numbers  $a_1, a_2, \dots, a_n \geq 0$  we have

$$(5.24) \quad (a_1 + \dots + a_n)^r \leq C_r \left( \sum_{1 \leq i \leq n} a_i^r + \sum_{1 \leq i_1 < \dots < i_r \leq n} a_{i_1} \cdot \dots \cdot a_{i_r} \right).$$

According to Lemma 5.5 and inequality (5.24) we have

$$\begin{aligned} &\left\| \left\| \mathcal{S}_{L \cup \{j_n\}, M}^{\sigma \oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell_{s_{j_n}}^2} \right\|_{\ell^{2r}}^{2r} \\ &= \sum_{x \in \mathbb{Z}^d} \left( \sum_{h \in \mathbb{N}_{\beta_{j_n}}} \sum_{w \in \mathcal{U}_{M^c}} |F_h^{\sigma, w}(x)|^2 \right)^r \\ &\lesssim_r \sum_{h \in \mathbb{N}_{\beta_{j_n}}} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} |F_h^{\sigma, w}(x)|^2 \right)^r \\ &\quad + \sum_{1 \leq h_1 < \dots < h_r \leq \beta_{j_n}} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} |F_{h_1}^{\sigma, w}(x)|^2 \right) \dots \left( \sum_{w \in \mathcal{U}_{M^c}} |F_{h_r}^{\sigma, w}(x)|^2 \right) \\ &= \sum_{s_{j_n} \in \mathbb{N}_{\beta_{j_n}}} \left\| \mathcal{S}_{L \cup \{j_n\}, M}^{\sigma \oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} \\ (5.25) \quad &+ \sum_{1 \leq h_1 < \dots < h_r \leq \beta_{j_n}} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} |F_{h_1}^{\sigma, w}(x)|^2 \right) \dots \left( \sum_{w \in \mathcal{U}_{M^c}} |F_{h_r}^{\sigma, w}(x)|^2 \right). \end{aligned}$$

The task now is to estimate the last double sum in (5.25). Recall that  $\mathcal{A}_{\beta_{j_n}} = \bigcup_{h=1}^{\beta_{j_n}} \mathcal{A}_{q_{j_n}, h}$  and note that for any  $\sigma \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}}$ , and  $(w_1, \dots, w_r) \in \mathcal{U}_{M^c}^r$  we have

$$F_h^{\sigma, w}(x) = \sum_{a \in \mathcal{A}_{\beta_{j_n}}} \mathcal{F}_{a/q_{j_n}, h}^{\sigma, w}(x).$$

Therefore,

$$\begin{aligned} &\sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} |F_{h_1}^{\sigma, w}(x)|^2 \right) \dots \left( \sum_{w \in \mathcal{U}_{M^c}} |F_{h_r}^{\sigma, w}(x)|^2 \right) \\ &= \sum_{(w_1, \dots, w_r) \in \mathcal{U}_{M^c}^r} \sum_{x \in \mathbb{Z}^d} |F_{h_1}^{\sigma, w_1}(x)|^2 \dots |F_{h_r}^{\sigma, w_r}(x)|^2 \\ (5.26) \quad &= \sum_{(w_1, \dots, w_r) \in \mathcal{U}_{M^c}^r} \sum_{(a_0(1), a_1(1), \dots, a_0(r), a_1(r)) \in \mathcal{A}_{\beta_{j_n}}^{2r}} \sum_{x \in \mathbb{Z}^d} \prod_{i=1}^r \mathcal{F}_{a_0(i)/q_{j_n}, h_i}^{\sigma, w_i}(x) \overline{\mathcal{F}_{a_1(i)/q_{j_n}, h_i}^{\sigma, w_i}(x)}. \end{aligned}$$

Likewise in the previous lemma let us introduce for any set  $Z \subseteq \mathcal{A}_{\beta_{j_n}}$  with  $|Z| \leq 2r$

$$(5.27) \quad S_Z(x) = \sum \mathcal{F}_{a_0(1)/q_{j_n}, h_1}^{\sigma, w_1}(x) \overline{\mathcal{F}_{a_1(1)/q_{j_n}, h_1}^{\sigma, w_1}(x)} \dots \mathcal{F}_{a_0(r)/q_{j_n}, h_r}^{\sigma, w_r}(x) \overline{\mathcal{F}_{a_1(r)/q_{j_n}, h_r}^{\sigma, w_r}(x)}$$

where the summation is taken over all sequences  $(a_0(1), a_1(1), \dots, a_0(r), a_1(r)) \in \mathcal{A}_{\beta_{j_n}}^{2r}$  which do not have the uniqueness property and  $\{a_0(1), a_1(1), \dots, a_0(r), a_1(r)\} = Z$ .



If the sequence  $(a_0(1), a_1(1), \dots, a_0(r), a_1(r)) \in \mathcal{A}_{\beta_{j_n}}^{2r}$  has the uniqueness property then (5.16) holds. Therefore,

$$(5.28) \quad \sum_{(a_0(1), a_1(1), \dots, a_0(r), a_1(r)) \in \mathcal{A}_{\beta_{j_n}}^{2r}} \sum_{x \in \mathbb{Z}^d} \prod_{i=1}^r \mathcal{F}_{a_0(i)/q_{j_n}, h_i}^{\sigma, w_i}(x) \overline{\mathcal{F}_{a_1(i)/q_{j_n}, h_i}^{\sigma, w_i}(x)} = \sum_{|Z| \leq r} \sum_{x \in \mathbb{Z}^d} S_Z(x).$$

The summation in the last sum is taken over all  $Z \subseteq \mathcal{A}_{\beta_{j_n}}$  such that  $|Z| \leq r$ , since by the uniqueness property,  $\sum_{x \in \mathbb{Z}^d} S_Z(x) = 0$  unless  $|Z| \leq r$ . By Lemma 5.4 we see that for each term in the sum from (5.27) we have

$$\prod_{i=1}^r |\mathcal{F}_{a_0(i)/q_{j_n}, h_i}^{\sigma, w_i}(x)| \cdot |\overline{\mathcal{F}_{a_1(i)/q_{j_n}, h_i}^{\sigma, w_i}(x)}| \leq 2 \sum_{\{a(1), \dots, a(r)\} = Z} \prod_{i=1}^r |\mathcal{F}_{a(i)/q_{j_n}, h_i}^{\sigma, w_i}(x)|^2.$$

Thus

$$(5.29) \quad \begin{aligned} \sum_{x \in \mathbb{Z}^d} \sum_{|Z| \leq r} S_Z(x) &\lesssim_r \sum_{x \in \mathbb{Z}^d} \sum_{|Z| \leq r} \sum_{\{a(1), \dots, a(r)\} = Z} \prod_{i=1}^r |\mathcal{F}_{a(i)/q_{j_n}, h_i}^{\sigma, w_i}(x)|^2 \\ &= \sum_{x \in \mathbb{Z}^d} \left( \sum_{a \in \mathcal{A}_{\beta_{j_n}}} |\mathcal{F}_{a/q_{j_n}, h_1}^{\sigma, w_1}(x)|^2 \right) \dots \left( \sum_{a \in \mathcal{A}_{\beta_{j_n}}} |\mathcal{F}_{a/q_{j_n}, h_r}^{\sigma, w_r}(x)|^2 \right) \end{aligned}$$

since for any  $Z \subseteq \mathbb{N}_{\beta_{j_n}}$  the sum in (5.27) contains at most  $C_r$  terms.

Combining (5.26), (5.28) and (5.29) we obtain that

$$\begin{aligned} &\sum_{1 \leq h_1 < \dots < h_r \leq \beta_{j_n}} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} |F_{h_1}^{\sigma, w}(x)|^2 \right) \dots \left( \sum_{w \in \mathcal{U}_{M^c}} |F_{h_r}^{\sigma, w}(x)|^2 \right) \\ &\lesssim_r \sum_{1 \leq h_1 < \dots < h_r \leq \beta_{j_n}} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \sum_{a \in \mathcal{A}_{\beta_{j_n}}} |\mathcal{F}_{a/q_{j_n}, h_1}^{\sigma, w}(x)|^2 \right) \dots \left( \sum_{w \in \mathcal{U}_{M^c}} \sum_{a \in \mathcal{A}_{\beta_{j_n}}} |\mathcal{F}_{a/q_{j_n}, h_r}^{\sigma, w}(x)|^2 \right) \\ &\lesssim_r \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \sum_{h \in \mathbb{N}_{\beta_{j_n}}} \sum_{a_h \in \mathcal{A}_{q_{j_n}, h}} \left| \sum_{u' \in \mathcal{U}_{M \setminus (L \cup \{j_n\})}} \sum_{v \in \mathcal{V}_L^c} f_{w+u'+v+a_h/q_{j_n}, h}(x) \right|^2 \right)^r \\ &= \|\mathcal{S}_{L, M \setminus \{j_n\}}^{\sigma}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r}. \end{aligned}$$

The proof of Lemma 5.6 is completed.  $\square$

Theorem 5.2 will be proved as a consequence of Theorem 5.3 and Proposition 5.1 below.

**Theorem 5.3.** *Suppose that  $\rho > 0$  and  $r \in \mathbb{N}$  are given. Then there is a constant  $C_{\rho, r} > 0$  such that for any  $N > 8^{2r/\rho}$  and for any set  $\Lambda$  as in (5.6) and for every  $f \in \ell^{2r}(\mathbb{Z}^d)$  we have*

$$(5.30) \quad \left\| \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u \right\|_{\ell^{2r}}^{2r} \leq C_{\rho, r} \sum_{\substack{M \subseteq \mathbb{N}_k \\ M = \{j_1, \dots, j_m\}}} \sum_{\sigma \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}} \|\mathcal{S}_{M, M}^{\sigma}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r}.$$

Moreover, the integer  $k \in \mathbb{N}_D$ , the set  $\mathcal{U}_{\mathbb{N}_k}$  and consequently the sets  $S_1, \dots, S_k$  are determined by the set  $\Lambda$  as it was described above the formulation of Theorem 5.2.

*Proof.* Observe that for any  $M \subseteq \mathbb{N}_k$  and  $L = \{j_1, \dots, j_l\} \subseteq M$  and  $j_n \in M \setminus L$  and for any  $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}}$  determined by the set  $L$ , by Lemma 5.5 combined with Lemma 5.6 we obtain that

$$(5.31) \quad \begin{aligned} \|\mathcal{S}_{L, M}^{\sigma}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} &\leq C_r \sum_{s_{j_n} \in \mathbb{N}_{\beta_{j_n}}} \|\mathcal{S}_{L \cup \{j_n\}, M}^{\sigma \oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \\ &\quad + C_r \|\mathcal{S}_{L, M \setminus \{j_n\}}^{\sigma}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r}. \end{aligned}$$

Therefore, applying (5.31) recursively we obtain

$$\begin{aligned}
\left\| \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u \right\|_{\ell^{2r}}^{2r} &= \|\mathcal{S}_{\emptyset, \mathbb{N}_k}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \\
&\lesssim_r \sum_{s_k \in \mathbb{N}_{\beta_{j_k}}} \|\mathcal{S}_{\{k\}, \mathbb{N}_k}^{(s_k)}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} + \|\mathcal{S}_{\emptyset, \mathbb{N}_{k-1}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \\
&\lesssim_r \sum_{s_{k-1} \in \mathbb{N}_{\beta_{j_{k-1}}}} \sum_{s_k \in \mathbb{N}_{\beta_{j_k}}} \|\mathcal{S}_{\{k-1, k\}, \mathbb{N}_k}^{(s_{k-1}, s_k)}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \\
&\quad + \sum_{s_k \in \mathbb{N}_{\beta_{j_k}}} \|\mathcal{S}_{\{k\}, \mathbb{N}_k \setminus \{k-1\}}^{(s_k)}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \\
&\quad + \sum_{s_{k-1} \in \mathbb{N}_{\beta_{j_{k-1}}}} \|\mathcal{S}_{\{k-1\}, \mathbb{N}_{k-1}}^{(s_{k-1})}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} + \|\mathcal{S}_{\emptyset, \mathbb{N}_{k-2}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \\
&\lesssim_r \cdots \lesssim_{\rho, r} \sum_{\substack{M \subseteq \mathbb{N}_k \\ M = \{j_1, \dots, j_m\}}} \sum_{\sigma \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \sum_{v \in \mathcal{V}_M^\sigma} f_{w+v}(x) \right|^2 \right)^r.
\end{aligned}$$

The proof of (5.30) is completed.  $\square$

**5.4. Concluding remarks and the proof of Theorem 5.2.** Now Theorem 5.3 reduces the proof of inequality (5.11) to showing

$$(5.32) \quad \sum_{\substack{M \subseteq \mathbb{N}_k \\ M = \{j_1, \dots, j_m\}}} \sum_{\sigma \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}} \|\mathcal{S}_{M, M}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \lesssim_{\rho, r} \|f\|_{\ell^{2r}}^{2r}$$

for any  $f \in \ell^{2r}(\mathbb{Z}^d)$  which is a characteristic function of a finite set in  $\mathbb{Z}^d$ . Firstly, we prove the following.

**Proposition 5.1.** *Under the assumptions of Theorem 5.2, there exists a constant  $C_{\rho, r} > 0$  such that for any  $M = \{j_1, \dots, j_m\} \subseteq \mathbb{N}_k$  any  $\sigma = (s_{j_1}, \dots, s_{j_m}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}$  determined by the set  $M$  and  $f \in \ell^{2r}(\mathbb{Z}^d)$  we have*

$$\|\mathcal{S}_{M, M}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}} \leq C_{\rho, r} \mathbf{A}_{2r} \left\| \mathcal{S}_{M, M}^\sigma \left( \mathcal{F}^{-1} \left( \sum_{b \in \mathbb{N}_{Q_0}} \eta_N(\xi - b/Q_0 - u) \hat{f}(\xi) \right) : u \in \mathcal{U}_{\mathbb{N}_k} \right) \right\|_{\ell^{2r}}.$$

*Proof.* We assume, without of loss of generality, that  $N \in \mathbb{N}$  is large. Let  $B_h = q_{j_1, s_{j_1}} \cdots q_{j_m, s_{j_m}} \cdot Q_0 \leq e^{N^\rho}$  and observe that according to the notation from (5.9), we have

$$\begin{aligned}
&\|\mathcal{S}_{M, M}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k})\|_{\ell^{2r}}^{2r} \\
&= \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \sum_{v \in \mathcal{V}_M^\sigma} f_{w+v}(x) \right|^2 \right)^r \\
&\leq \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \mathcal{F}^{-1} \left( \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} \Theta(\xi - b/Q_0 - v - w) \eta_N(\xi - b/Q_0 - v - w) \hat{f}(\xi) \right) (x) \right|^2 \right)^r \\
&= \sum_{n \in \mathbb{N}_{B_h}^d} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \mathcal{F}^{-1}(\Theta \eta_N G(\cdot; n, w))(B_h x + n) \right|^2 \right)^r
\end{aligned}$$

where

$$G(\xi; n, w) = \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}^d} \hat{f}(\xi + b/Q_0 + v + w) e^{-2\pi i(b/Q_0 + v) \cdot n}.$$

By Proposition 4.2, with the sequence of multipliers  $\Theta_N = \Theta$  for all  $N \in \mathbb{N}$  and  $\Theta$  as in (5.2), we have

$$(5.33) \quad \left\| \mathcal{F}^{-1}(\Theta \eta_N G(\cdot; n, w))(B_h x + n) \right\|_{\ell^{2r}(x)} \leq C_{\rho, r} \mathbf{A}_{2r} \left\| \mathcal{F}^{-1}(\eta_N G(\cdot; n, w))(B_h x + n) \right\|_{\ell^{2r}(x)}$$

since  $\inf_{\gamma \in \Gamma} \varepsilon_\gamma^{-1} \geq e^{N^{2\rho}} \geq 2e^{(d+1)N^\rho} \geq B_h$  for sufficiently large  $N \in \mathbb{N}$ . Therefore, combining (5.33) with (4.6) we obtain that

$$\begin{aligned} & \sum_{n \in \mathbb{N}_{B_h}^d} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} |\mathcal{F}^{-1}(\Theta \eta_N G(\cdot; n, w))(B_h x + n)|^2 \right)^r \\ & \lesssim_r \sum_{n \in \mathbb{N}_{B_h}^d} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} |\mathcal{F}^{-1}(\eta_N G(\cdot; n, w))(B_h x + n)|^2 \right)^r \\ & = \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \mathcal{F}^{-1} \left( \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} \eta_N(\xi - b/Q_0 - v - w) \hat{f}(\xi) \right)(x) \right|^2 \right)^r. \end{aligned}$$

This completes the proof of Proposition 5.1.  $\square$

Now we are able to finish the proof of Theorem 5.2.

*Proof of Theorem 5.2.* It remains to show that there exists a constant  $C_{\rho, r} > 0$  such that for any  $M = \{j_1, \dots, j_m\} \subseteq \mathbb{N}_k$  any  $\sigma = (s_{j_1}, \dots, s_{j_m}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}$  determined by the set  $M$  and  $f \in \ell^{2r}(\mathbb{Z}^d)$  we have

$$\sum_{\sigma \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}} \left\| \mathcal{S}_{M, M}^\sigma \left( \mathcal{F}^{-1} \left( \sum_{b \in \mathbb{N}_{Q_0}} \eta_N(\xi - b/Q_0 - u) \hat{f}(\xi) \right) : u \in \mathcal{U}_{\mathbb{N}_k} \right) \right\|_{\ell^{2r}}^{2r} \leq C_{\rho, r}^{2r} \|f\|_{\ell^{2r}}^{2r}.$$

Since there are  $2^k$  possible choices of sets  $M \subseteq \mathbb{N}_k$  and  $k \in \mathbb{N}_D$  then (5.32) will follow and the proof of Theorem 5.2 will be completed. If  $r = 1$  then Plancherel's theorem does the job since the functions  $\eta_N(\xi - b/Q_0 - v - w)$  are disjointly supported for all  $b/Q_0 \in \mathbb{N}_{Q_0}$ ,  $w \in \mathcal{U}_{M^c}$ ,  $v \in \mathcal{V}_M^\sigma$  and  $\sigma = (s_{j_1}, \dots, s_{j_m}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}$ . For general  $r \geq 2$ , since  $\|f\|_{\ell^{2r}}^{2r} = \|f\|_{\ell^2}^2$  because we have assumed that  $f$  is a characteristic function of a finite set in  $\mathbb{Z}^d$ , it suffices to prove for any  $x \in \mathbb{Z}^d$  that

$$\sum_{w \in \mathcal{U}_{M^c}} \left| \mathcal{F}^{-1} \left( \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} \eta_N(\xi - b/Q_0 - v - w) \hat{f}(\xi) \right)(x) \right|^2 \leq C_{\rho, r}.$$

In fact, since  $\|f\|_{\ell^\infty} = 1$ , it is enough to show

$$(5.34) \quad \left\| \mathcal{F}^{-1} \left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} \eta_N(\xi - b/Q_0 - v - w) \right) \right\|_{\ell^1} \leq C_{\rho, r}$$

for any sequence of complex numbers  $(\alpha(w) : w \in \mathcal{U}_{M^c})$  such that

$$(5.35) \quad \sum_{w \in \mathcal{U}_{M^c}} |\alpha(w)|^2 = 1.$$

Computing the Fourier transform we obtain

$$\begin{aligned} & \mathcal{F}^{-1} \left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} \eta_N(\xi - b/Q_0 - v - w) \right)(x) \\ & = \left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) e^{-2\pi i x \cdot w} \right) \cdot \det(\mathcal{E}_N) \mathcal{F}^{-1} \eta(\mathcal{E}_N x) \cdot \left( \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} e^{-2\pi i x \cdot (b/Q_0 + v)} \right). \end{aligned}$$

If  $v = a_{j_1, s_{j_1}}/q_{j_1, s_{j_1}} + v'$  for some  $v' \in V_{M \setminus \{j_1\}}^\tau$  where  $\tau = (s_{j_2}, \dots, s_{j_m})$  then

$$\begin{aligned} (5.36) \quad & \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} e^{-2\pi i x \cdot (b/Q_0 + v)} = \sum_{v' \in \mathcal{V}_{M \setminus \{j_1\}}^\tau} \sum_{b \in \mathbb{N}_{Q_1}} e^{-2\pi i x \cdot (b/Q_1 + v')} \\ & - \sum_{v' \in \mathcal{V}_{M \setminus \{j_1\}}^\tau} \sum_{b \in \mathbb{N}_{Q_2}} e^{-2\pi i x \cdot (b/Q_2 + v')} \end{aligned}$$

where  $q_{j_1, s_{j_1}} = p_{j_1, s_{j_1}}^{\gamma_1}$ ,  $Q_1 = Q_0 p_{j_1, s_{j_1}}^{\gamma_1}$  and  $Q_2 = Q_0 p_{j_1, s_{j_1}}^{\gamma_1 - 1}$ . The identity (5.36) shows that the function on the left hand side of (5.36) can be written as a sum of  $2^m$  functions

$$\sum_{b \in \mathbb{N}_Q} e^{-2\pi i x \cdot (b/Q)} = \begin{cases} Q^d & \text{if } x \equiv 0 \pmod{Q}, \\ 0 & \text{otherwise,} \end{cases}$$

where possible values of  $Q$  are products of  $Q_0$  and  $p_{j_i, s_{j_i}}^{\gamma_i}$  or  $p_{j_i, s_{j_i}}^{\gamma_i-1}$  for  $i \in \mathbb{N}_m$ . Therefore, the proof of (5.34) will be completed if we show that

$$\left\| \left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) e^{-2\pi i Q x \cdot w} \right) \cdot Q^d \det(\mathcal{E}_N) \mathcal{F}^{-1} \eta(Q \mathcal{E}_N x) \right\|_{\ell^1(x)} \leq C_{\rho, r}$$

for any integer  $Q \leq e^{N^\rho}$  such that  $(Q, q_{j,s}) = 1$ , for all  $j \in M^c$  and  $s \in \mathbb{N}_{\beta_j}$ .

Recall that, according Remark 4.1, in our case  $\eta = \phi * \psi$  for some two smooth functions  $\phi, \psi$  supported in  $(-1/(8d), 1/(8d))^d$ . Therefore, by the Cauchy–Schwarz inequality we only need to prove that

$$(5.37) \quad Q^{d/2} \det(\mathcal{E}_N)^{1/2} \left\| \mathcal{F}^{-1} \phi(Q \mathcal{E}_N x) \right\|_{\ell^2(x)} \leq C_{\rho, r}$$

and

$$(5.38) \quad Q^{d/2} \det(\mathcal{E}_N)^{1/2} \left\| \left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) e^{-2\pi i Q x \cdot w} \right) \cdot \mathcal{F}^{-1} \psi(Q \mathcal{E}_N x) \right\|_{\ell^2(x)} \leq C_{\rho, r}.$$

Since  $(Q, q_{j,s}) = 1$ , for all  $j \in M^c$  and  $s \in \mathbb{N}_{\beta_j}$  then  $Qw \notin \mathbb{Z}^d$  for any  $w \in \mathcal{U}_{M^c}$  and its denominator is bounded by  $N^D$ . We can assume, without of loss of generality, that  $Qw \in [0, 1)^d$  by the periodicity of  $x \mapsto e^{-2\pi i x \cdot Qw}$ . Inequality (5.37) easily follows from Plancherel's theorem. In order to prove (5.38) observe that by the change of variables one has

$$\left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) e^{-2\pi i x \cdot Qw} \right) \cdot \mathcal{F}^{-1} \psi(Q \mathcal{E}_N x) = Q^{-d} \det(\mathcal{E}_N)^{-1} \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) \mathcal{F}^{-1} (\psi(Q^{-1} \mathcal{E}_N^{-1}(\cdot - Qw)))(x).$$

Therefore, Plancherel's theorem and the last identity yield

$$(5.39) \quad \begin{aligned} & Q^d \det(\mathcal{E}_N) \left\| \left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) e^{-2\pi i Q x \cdot w} \right) \cdot \mathcal{F}^{-1} \psi(Q \mathcal{E}_N x) \right\|_{\ell^2(x)}^2 \\ &= \sum_{w \in \mathcal{U}_{M^c}} |\alpha(w)|^2 \int_{\mathbb{R}^d} |\psi(\xi - \mathcal{E}_N^{-1} w)|^2 d\xi \\ &+ \sum_{\substack{w_1, w_2 \in \mathcal{U}_{M^c} \\ w_1 \neq w_2}} \alpha(w_1) \overline{\alpha(w_2)} \int_{\mathbb{R}^d} \psi(\xi) \overline{\psi(\xi - \mathcal{E}_N^{-1}(w_1 - w_2))} d\xi. \end{aligned}$$

The first sum on the right-hand side of (5.39) is bounded in view of (5.35). The second one vanishes since the function  $\psi$  is supported in  $(-1/(8d), 1/(8d))^d$  and  $|\mathcal{E}_N^{-1}(w_1 - w_2)|_\infty \geq e^{N^{2\rho}} N^{-2D} > 1$ , for sufficiently large  $N$ . The proof of Theorem 5.2 is completed.  $\square$

## 6. VECTOR-VALUED MAXIMAL ESTIMATES FOR AVERAGING OPERATORS

For any function  $f : \mathbb{Z}^d \rightarrow \mathbb{C}$  with a finite support we have

$$M_N f(x) = K_N * f(x)$$

where  $K_N$  is a kernel defined by

$$(6.1) \quad K_N(x) = N^{-k} \sum_{y \in \mathbb{N}_N^k} \delta_{\mathcal{Q}(y)}$$

and  $\delta_y$  denotes Dirac's delta at  $y \in \mathbb{Z}^k$  and  $\mathcal{Q}$  is the canonical polynomial, see Section 2. Let  $m_N$  denote the discrete Fourier transform of  $K_N$ , i.e.

$$m_N(\xi) = N^{-k} \sum_{y \in \mathbb{N}_N^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)}.$$

Finally, we define

$$\Phi_N(\xi) = \int_{[0,1]^k} e^{2\pi i \xi \cdot \mathcal{Q}(Ny)} dy.$$

Using a multi-dimensional version of van der Corput lemma (see [15, 4, 17]) we may estimate

$$(6.2) \quad |\Phi_N(\xi)| \lesssim \min \{1, |N^A \xi|_\infty^{-1/d}\}$$

where  $A$  is the  $d \times d$  diagonal matrix as in Section 2. Additionally, we have

$$(6.3) \quad |\Phi_N(\xi) - 1| \lesssim \min \{1, |N^A \xi|_\infty\}.$$

For  $q \in \mathbb{N}$  let us define

$$A_q = \{a \in \mathbb{N}_q^d : \gcd(q, (a_\gamma : \gamma \in \Gamma))\}.$$

Next, for  $q \in \mathbb{N}$  and  $a \in A_q$  we define the *Gaussian sum*

$$G(a/q) = q^{-k} \sum_{y \in \mathbb{N}_q^k} e^{2\pi i(a/q) \cdot \mathcal{Q}(y)}.$$

Let us observe that, by the multi-dimensional variant of Weyl's inequality (see [16, Proposition 3]), there exists  $\delta > 0$  such that

$$(6.4) \quad |G(a/q)| \lesssim q^{-\delta}.$$

We shall prove that for every  $p \in (1, \infty)$  there is a constant  $C_p > 0$  such that for every sequence  $(f_t : t \in \mathbb{N}) \in \ell^p(\ell^2(\mathbb{Z}^d))$  of non-negative functions with finite supports we have

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} |M_{2^n} f_t|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

We begin with proving the following.

**Proposition 6.1.** *There is a constant  $C > 0$  such that for every  $N \in \mathbb{N}$  and for every  $\xi \in [-1/2, 1/2]^d$  satisfying*

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \leq L_1^{-|\gamma|} L_2$$

for all  $\gamma \in \Gamma$ , where  $1 \leq q \leq L_3 \leq N^{1/2}$ ,  $a \in A_q$ ,  $L_1 \geq N$  and  $L_2 \geq 1$  we have

$$|m_N(\xi) - G(a/q) \Phi_N(\xi - a/q)| \leq C \left( L_3 N^{-1} + L_2 L_3 N^{-1} \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|} \right) \lesssim L_2 L_3 N^{-1}.$$

*Proof.* Let  $\theta = \xi - a/q$ . For any  $r \in \mathbb{N}_q^k$ , if  $y \equiv r \pmod{q}$  then for each  $\gamma \in \Gamma$

$$\xi_\gamma y^\gamma \equiv \theta_\gamma y^\gamma + (a_\gamma/q) r^\gamma \pmod{1},$$

thus

$$\xi \cdot \mathcal{Q}(y) \equiv \theta \cdot \mathcal{Q}(y) + (a/q) \cdot \mathcal{Q}(r) \pmod{1}.$$

Therefore,

$$N^{-k} \sum_{y \in \mathbb{N}_N^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} = q^{-k} \sum_{r \in \mathbb{N}_q^k} e^{2\pi i(a/q) \cdot \mathcal{Q}(r)} \cdot \left( q^k N^{-k} \sum_{\substack{y \in \mathbb{N}_N^k \\ qy+r \in [1, N]^k}} e^{2\pi i \theta \cdot \mathcal{Q}(qy+r)} \right).$$

If  $|qy + r|, |qy| \leq N$  then

$$|\theta \cdot \mathcal{Q}(qy + r) - \theta \cdot \mathcal{Q}(qy)| \lesssim |r| \sum_{\gamma \in \Gamma} |\theta_\gamma| \cdot N^{(|\gamma|-1)} \lesssim q \sum_{\gamma \in \Gamma} L_1^{-|\gamma|} L_2 N^{(|\gamma|-1)} \lesssim L_2 L_3 N^{-1} \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|}.$$

Thus

$$N^{-k} \sum_{y \in \mathbb{N}_N^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} = G(a/q) \cdot q^k N^{-k} \sum_{\substack{y \in \mathbb{N}_N^k \\ qy \in [1, N]^k}} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} + \mathcal{O} \left( q N^{-1} + L_2 L_3 N^{-1} \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|} \right).$$

Again by the mean value theorem one can replace the sum on the right-hand side by the integral. Indeed, we obtain

$$\begin{aligned} & \left| \sum_{y \in (0, [N/q]^k)} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} - \int_{[0, N/q]^k} e^{2\pi i \theta \cdot \mathcal{Q}(qt)} dt \right| \\ &= \left| \sum_{y \in (0, [N/q]^k)} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} - \sum_{y \in (0, [N/q]-1]^k} \int_{y+(0, 1]^k} e^{2\pi i \theta \cdot \mathcal{Q}(qt)} dt \right| + \mathcal{O}((N/q)^{k-1}) \\ &= \left| \sum_{y \in (0, [N/q]-1]^k} \int_{(0, 1]^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} - e^{2\pi i \theta \cdot \mathcal{Q}(q(t+y))} dt \right| + \mathcal{O}((N/q)^{k-1}) \\ &= \mathcal{O} \left( (N/q)^{k-1} + (N/q)^k L_2 L_3 N^{-1} \sum_{\gamma \in \Gamma} (N/L_1)^{|\gamma|} \right). \end{aligned}$$

This completes the proof of Proposition 6.1. □

Let  $\chi > 0$  and  $l \in \mathbb{N}$  be the numbers whose precise values will be specified later. Let us introduce for every  $n \in \mathbb{N}_0$  the multipliers

$$\Xi_n(\xi) = \sum_{a/q \in \mathcal{U}_{n^l}} \eta(2^{n(A-\chi I)}(\xi - a/q))$$

with  $\mathcal{U}_{n^l}$  defined as in (5.1). From Theorem 5.1 we know that for every  $p \in (1, \infty)$

$$(6.5) \quad \|\mathcal{F}^{-1}(\Xi_n \hat{f})\|_{\ell^p} \lesssim \log(n+2) \|f\|_{\ell^p}.$$

The implicit constant in (6.5) depends on the parameter  $\rho > 0$ , which was fixed, see Section 5. However, from now on we will assume that  $\rho > 0$  and the integer  $l \geq 10$  are related by the equation

$$(6.6) \quad 10\rho l = 1.$$

Observe that

$$(6.7) \quad \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} |M_{2^n} f_t|^2 \right)^{1/2} \right\|_{\ell^p} \leq \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} |\mathcal{F}^{-1}(m_{2^n} \Xi_n \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} + \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} |\mathcal{F}^{-1}(m_{2^n} (1 - \Xi_n) \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}.$$

**6.1. The estimate for the second norm in (6.7).** Replacing the supremum norm by  $\ell^2$  norm we see that

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} |\mathcal{F}^{-1}(m_{2^n} (1 - \Xi_n) \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \leq \sum_{n \in \mathbb{N}_0} \left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{F}^{-1}(m_{2^n} (1 - \Xi_n) \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}.$$

Therefore, it suffices to show

$$\left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{F}^{-1}(m_{2^n} (1 - \Xi_n) \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim (n+1)^{-2} \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}$$

which in view of (4.6) holds, if we prove

$$(6.8) \quad \|\mathcal{F}^{-1}(m_{2^n} (1 - \Xi_n) \hat{f})\|_{\ell^p} \lesssim (n+1)^{-2} \|f\|_{\ell^p}.$$

Indeed, by (6.5) we have for every  $1 < p < \infty$  that

$$(6.9) \quad \|\mathcal{F}^{-1}(m_{2^n} (1 - \Xi_n) \hat{f})\|_{\ell^p} \leq \|M_{2^n} f\|_{\ell^p} + \|M_{2^n} (\mathcal{F}^{-1}(\Xi_n \hat{f}))\|_{\ell^p} \lesssim \log(n+2) \|f\|_{\ell^p}.$$

In fact we show that it is possible to improve the estimate (6.9) for  $p = 2$ . Namely, we will show that for big enough  $\alpha > 0$ , which will be specified later, and for all  $n \in \mathbb{N}_0$  we have

$$(6.10) \quad |m_{2^n}(\xi)(1 - \Xi_n(\xi))| \lesssim (n+1)^{-\alpha}.$$

This estimate will be a consequence of Theorem 3.1. For do so, by Dirichlet's principle we have for every  $\gamma \in \Gamma$

$$|\xi_\gamma - a_\gamma/q_\gamma| \leq q_\gamma^{-1} n^\beta 2^{-n|\gamma|}$$

where  $1 \leq q_\gamma \leq n^{-\beta} 2^{n|\gamma|}$ . In order to apply Theorem 3.1 we must show that there exists  $\gamma \in \Gamma$  such that  $n^\beta \leq q_\gamma \leq n^{-\beta} 2^{n|\gamma|}$ . Suppose for a contradiction that for every  $\gamma \in \Gamma$  we have  $1 \leq q_\gamma < n^\beta$  then for some  $q \leq \text{lcm}(q_\gamma : \gamma \in \Gamma) \leq n^{\beta d}$  we have

$$|\xi_\gamma - a'_\gamma/q| \leq n^\beta 2^{-n|\gamma|}$$

where  $\gcd(q, \gcd(a'_\gamma : \gamma \in \Gamma)) = 1$ . Hence, taking  $a' = (a'_\gamma : \gamma \in \Gamma)$  we have  $a'/q \in \mathcal{U}_{n^l}$  provided that  $\beta d < l$ . On the other hand, if  $1 - \Xi_n(\xi) \neq 0$  then for every  $a'/q \in \mathcal{U}_{n^l}$  there exists  $\gamma \in \Gamma$  such that

$$|\xi_\gamma - a'_\gamma/q| > 2^{-n(|\gamma|-\chi)}/16d.$$

Therefore, one obtains

$$2^{\chi n} < 16dn^\beta$$

but this gives a contradiction, for sufficiently large  $n \in \mathbb{N}$ . We have already shown that there exists  $\gamma \in \Gamma$  such that  $n^\beta \leq q_\gamma \leq n^{-\beta} 2^{n|\gamma|}$  and consequently Theorem 3.1 yields

$$|m_{2^n}(\xi)| \lesssim (n+1)^{-\alpha}$$

provided that  $1 - \Xi_n(\xi) \neq 0$  and this proves (6.10) and we obtain

$$(6.11) \quad \|\mathcal{F}^{-1}(m_{2^n} (1 - \Xi_n) \hat{f})\|_{\ell^2} \lesssim (1+n)^{-\alpha} \log(n+2) \|f\|_{\ell^2}.$$

Interpolating (6.11) with (6.9) we obtain

$$\|\mathcal{F}^{-1}(m_{2^n}(1 - \Xi_n)\hat{f})\|_{\ell^p} \lesssim (1+n)^{-c_p\alpha} \log(n+2) \|f\|_{\ell^p}.$$

for some  $c_p > 0$ . Choosing  $\alpha > 0$  and  $l \in \mathbb{N}$  appropriately large one obtains (6.8).

**6.2. The estimate for the first norm in (6.7).** Note that for any  $\xi \in \mathbb{T}^d$  so that

$$|\xi_\gamma - a_\gamma/q| \leq (8d)^{-1} 2^{-n(|\gamma|-\chi)}$$

for every  $\gamma \in \Gamma$  with  $1 \leq q \leq e^{n^{1/10}}$  we have

$$(6.12) \quad m_{2^n}(\xi) = G(a/q)\Phi_{2^n}(\xi - a/q) + q^{-\delta}E_{2^n}(\xi)$$

where

$$(6.13) \quad |E_{2^n}(\xi)| \lesssim 2^{-n/2}.$$

These two properties (6.12) and (6.13) follow from Proposition 6.1 with  $L_1 = 2^n$ ,  $L_2 = 2^{\chi n}$  and  $L_3 = e^{n^{1/10}}$ , since

$$|E_{2^n}(\xi)| \lesssim q^\delta L_2 L_3 2^{-n} \lesssim (e^{-n((1-\chi)\log 2 - 2n^{-9/10})}) \lesssim 2^{-n/2}$$

which holds for sufficiently large  $n \in \mathbb{N}$ , when  $\chi > 0$  is sufficiently small. Let us introduce for every  $n \in \mathbb{N}_0$  new multipliers

$$\nu_{2^n}(\xi) = \sum_{a/q \in \mathcal{U}_{n^l}} G(a/q)\Phi_{2^n}(\xi - a/q)\eta(2^{n(A-\chi I)}(\xi - a/q))$$

and note that by (6.12)

$$|m_{2^n}(\xi)\Xi_n(\xi) - \nu_{2^n}(\xi)| \lesssim 2^{-n/2}$$

and consequently by Plancherel's theorem

$$(6.14) \quad \|\mathcal{F}^{-1}((m_{2^n}\Xi_n - \nu_{2^n})\hat{f})\|_{\ell^2} \lesssim 2^{-n/2} \|f\|_{\ell^2}.$$

Moreover, by Theorem 5.1 we have

$$\|\mathcal{F}^{-1}(m_{2^n}\Xi_n\hat{f})\|_{\ell^p} \lesssim \log(n+2) \|f\|_{\ell^p}$$

and

$$\|\mathcal{F}^{-1}(\nu_{2^n}\hat{f})\|_{\ell^p} \lesssim |\mathcal{U}_{n^l}| \cdot \|f\|_{\ell^p} \lesssim e^{(d+1)n^{1/10}} \|f\|_{\ell^p}$$

thus

$$(6.15) \quad \|\mathcal{F}^{-1}((m_{2^n}\Xi_n - \nu_{2^n})\hat{f})\|_{\ell^p} \lesssim e^{(d+1)n^{1/10}} \|f\|_{\ell^p}.$$

Interpolating now (6.14) with (6.15) we can conclude that for some  $c_p > 0$

$$(6.16) \quad \|\mathcal{F}^{-1}((m_{2^n}\Xi_n - \nu_{2^n})\hat{f})\|_{\ell^p} \lesssim 2^{-c_p n} \|f\|_{\ell^p}.$$

For every  $n, s \in \mathbb{N}_0$  define multipliers

$$\nu_{2^n}^s(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} G(a/q)\Phi_{2^n}(\xi - a/q)\eta(2^{s(A-\chi I)}(\xi - a/q))$$

and note that by (6.2) we have

$$(6.17) \quad \begin{aligned} & |\nu_{2^n}(\xi) - \sum_{0 \leq s < n} \nu_{2^n}^s(\xi)| \\ & \leq \sum_{0 \leq s < n} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} |G(a/q)| |\Phi_{2^n}(\xi - a/q)| |\eta(2^{s(A-\chi I)}(\xi - a/q)) - \eta(2^{n(A-\chi I)}(\xi - a/q))| \\ & \lesssim 2^{-\chi n/d} \end{aligned}$$

since  $|\Phi_{2^n}(\xi - a/q)| \lesssim 2^{-\chi n/d}$ , provided that  $\eta(2^{s(A-\chi I)}(\xi - a/q)) - \eta(2^{n(A-\chi I)}(\xi - a/q)) \neq 0$ . The estimate (6.17) combined with Plancherel's theorem implies that

$$(6.18) \quad \|\mathcal{F}^{-1}((\nu_{2^n} - \sum_{0 \leq s < n} \nu_{2^n}^s)\hat{f})\|_{\ell^2} \lesssim 2^{-\chi n/d} \|f\|_{\ell^2}$$



Moreover, since  $|\mathcal{U}_{s^l}| \leq |\mathcal{U}_{n^l}| \lesssim e^{(d+1)n^{1/10}}$  we have

$$(6.19) \quad \left\| \mathcal{F}^{-1} \left( \left( \nu_{2^n} - \sum_{0 \leq s < n} \nu_{2^n}^s \right) \hat{f} \right) \right\|_{\ell^p} \lesssim e^{(d+1)n^{1/10}} \|f\|_{\ell^p}.$$

Interpolating (6.18) with (6.19) one immediately concludes that for some  $c_p > 0$

$$(6.20) \quad \left\| \mathcal{F}^{-1} \left( \left( \nu_{2^n} - \sum_{0 \leq s < n} \nu_{2^n}^s \right) \hat{f} \right) \right\|_{\ell^p} \lesssim 2^{-c_p n} \|f\|_{\ell^p}.$$

In view of (4.6) and (6.16) and (6.20) it suffices to prove that for every  $s \in \mathbb{N}_0$  we have

$$(6.21) \quad \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} |\mathcal{F}^{-1}(\nu_{2^n}^s \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim (s+1)^{-2} \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

**6.3.  $\ell^2(\mathbb{Z}^d)$  estimates for (6.21).** Our aim will be to prove

$$(6.22) \quad \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} |\mathcal{F}^{-1}(\nu_{2^n}^s \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^2} \lesssim (s+1)^{-\delta l+1} \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^2}.$$

For do so, we only need to prove the following.

**Theorem 6.1.** *There is  $C > 0$  such that for any  $s \in \mathbb{N}_0$  and  $f \in \ell^2(\mathbb{Z}^d)$*

$$\left\| \sup_{n \in \mathbb{N}_0} |\mathcal{F}^{-1}(\nu_{2^n}^s \hat{f})| \right\|_{\ell^2} \leq C(s+1)^{-\delta l+1} \|f\|_{\ell^2}$$

with  $l \in \mathbb{N}$  defined in (6.6) and  $\delta > 0$  as in (6.4).

*Proof.* For  $s \in \mathbb{N}_0$  we set  $\kappa_s = 20d(\lfloor (s+1)^{1/10} \rfloor + 1)$  and  $Q_s = (\lfloor e^{(s+1)^{1/10}} \rfloor)!$ . Firstly, we estimate the supremum over  $0 \leq n \leq 2^{\kappa_s}$ . By Lemma 2.2 we have

$$\left\| \sup_{0 \leq n \leq 2^{\kappa_s}} |\mathcal{F}^{-1}(\nu_{2^n}^s \hat{f})| \right\|_{\ell^2} \lesssim \left\| \mathcal{F}^{-1}(\nu_1^s \hat{f}) \right\|_{\ell^2} + \sum_{i=0}^{\kappa_s} \left( \sum_{j=0}^{2^{\kappa_s-i}-1} \left\| \sum_{m \in I_j^i} \mathcal{F}^{-1}((\nu_{2^{m+1}}^s - \nu_{2^m}^s) \hat{f}) \right\|_{\ell^2}^2 \right)^{1/2}$$

where  $I_j^i = [j2^i, (j+1)2^i)$ . For any  $i \in \{0, \dots, \kappa_s\}$ , by Plancherel's theorem we get

$$\begin{aligned} \sum_{j=0}^{2^{\kappa_s-i}-1} \left\| \sum_{m \in I_j^i} \mathcal{F}^{-1}((\nu_{2^{m+1}}^s - \nu_{2^m}^s) \hat{f}) \right\|_{\ell^2}^2 &\leq \sum_{a/q \in \mathcal{U}_{(s+1)^i} \setminus \mathcal{U}_{s^i}} |G(a/q)|^2 \\ &\times \sum_{j=0}^{2^{\kappa_s-i}-1} \sum_{m, m' \in I_j^i} \int_{\mathbb{T}^d} |\Delta_m(\xi - a/q)| \cdot |\Delta_{m'}(\xi - a/q)| \cdot \eta_s(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi, \end{aligned}$$

where  $\Delta_m(\xi) = \Phi_{2^{m+1}}(\xi) - \Phi_{2^m}(\xi)$  and  $\eta_s(\xi) = \eta(2^{s(A-\chi I)}\xi)$ , since the supports are effectively disjoint. Using (6.2) and (6.3) we conclude

$$\sum_{m \in \mathbb{Z}} |\Delta_m(\xi)| \lesssim \sum_{m \in \mathbb{Z}} \min \{ |2^{mA} \xi|_\infty, |2^{mA} \xi|^{-1/d} \} \lesssim 1.$$

Therefore, by (6.4) we may estimate

$$\begin{aligned} \sum_{j=0}^{2^{\kappa_s-i}-1} \left\| \sum_{m \in I_j^i} \mathcal{F}^{-1}((\nu_{2^{m+1}}^s - \nu_{2^m}^s) \hat{f}) \right\|_{\ell^2}^2 &\lesssim (s+1)^{-2\delta l} \sum_{a/q \in \mathcal{U}_{(s+1)^i} \setminus \mathcal{U}_{s^i}} \int_{\mathbb{T}^d} \eta_s(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \\ &\lesssim (s+1)^{-2\delta l} \|f\|_{\ell^2}^2 \end{aligned}$$

since if  $a/q \in \mathcal{U}_{(s+1)^i} \setminus \mathcal{U}_{s^i}$  then  $q \geq s^l$ . In the last step we have used disjointness of supports of  $\eta_s(\cdot - a/q)$  while  $a/q$  varies over  $\mathcal{U}_{(s+1)^i} \setminus \mathcal{U}_{s^i}$ . We have just proven

$$(6.23) \quad \left\| \sup_{0 \leq n \leq 2^{\kappa_s}} |\mathcal{F}^{-1}(\nu_{2^n}^s \hat{f})| \right\|_{\ell^2} \lesssim \kappa_s (s+1)^{-\delta l} \|f\|_{\ell^2} \lesssim (s+1)^{-\delta l+1} \|f\|_{\ell^2}.$$

Next, we consider the case when the supremum is taken over  $n \geq 2^{\kappa_s}$ . For any  $x, y \in \mathbb{Z}^d$  we define

$$I(x, y) = \sup_{n \geq 2^{\kappa_s}} \left| \sum_{a/q \in \mathcal{U}_{(s+1)^i} \setminus \mathcal{U}_{s^i}} G(a/q) e^{-2\pi i(a/q) \cdot x} \mathcal{F}^{-1}(\Phi_{2^n} \eta_s \hat{f}(\cdot + a/q))(y) \right|$$

and

$$J(x, y) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} G(a/q) e^{-2\pi i(a/q) \cdot x} \mathcal{F}^{-1}(\eta_s \hat{f}(\cdot + a/q))(y).$$

By Plancherel's theorem, for any  $u \in \mathbb{N}_{Q_s}^d$  and  $a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$  we have

$$\begin{aligned} & \left\| \mathcal{F}^{-1}(\Phi_{2^n} \eta_s \hat{f}(\cdot + a/q))(x + u) - \mathcal{F}^{-1}(\Phi_{2^n} \eta_s \hat{f}(\cdot + a/q))(x) \right\|_{\ell^2(x)} \\ &= \left\| (1 - e^{-2\pi i \xi \cdot u}) \Phi_{2^n}(\xi) \eta_s(\xi) \hat{f}(\xi + a/q) \right\|_{L^2(d\xi)} \\ &\lesssim 2^{-n/d} \cdot |u| \cdot \left\| \eta_s(\cdot - a/q) \hat{f} \right\|_{L^2} \end{aligned}$$

since, by (6.2),

$$\sup_{\xi \in \mathbb{T}^d} |\xi| \cdot |\Phi_{2^n}(\xi)| \lesssim \sup_{\xi \in \mathbb{T}^d} |\xi| \cdot |2^{nA} \xi|^{-1/d} \leq 2^{-n/d}.$$

Therefore,

$$\left| \|I(x, x + u)\|_{\ell^2(x)} - \|I(x, x)\|_{\ell^2(x)} \right| \leq |u| \sum_{n=2^{\kappa_s}}^{\infty} 2^{-n/d} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} \|\eta_s(\cdot - a/q) \hat{f}\|_{\ell^2}$$

because the set  $\mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l} \subseteq \mathcal{U}_{(s+1)^l}$  contains at most  $e^{(d+1)(s+1)^{1/10}}$  elements and

$$2^{\kappa_s} (\log 2)/d - (s+1)^{1/10} e^{(s+1)^{1/10}} - (d+1)(s+1)^{1/10} \geq s$$

for sufficiently large  $s \geq 0$ . Thus we obtain

$$\|I(x, x)\|_{\ell^2(x)} \lesssim \|I(x, x + u)\|_{\ell^2(x)} + 2^{-s} \|f\|_{\ell^2}.$$

In particular,

$$\left\| \sup_{n \geq 2^{\kappa_s}} |\mathcal{F}^{-1}(\nu_{2^n}^s \hat{f})| \right\|_{\ell^2}^2 \lesssim \frac{1}{Q_s^d} \sum_{u \in \mathbb{N}_{Q_s}^d} \|I(x, x + u)\|_{\ell^2(x)}^2 + 2^{-2s} \|f\|_{\ell^2}^2.$$

Let us observe that the functions  $x \mapsto I(x, y)$  and  $x \mapsto J(x, y)$  are  $Q_s \mathbb{Z}^d$ -periodic. Therefore, by double change of variables we get

$$\sum_{u \in \mathbb{N}_{Q_s}^d} \|I(x, x + u)\|_{\ell^2(x)}^2 = \sum_{x \in \mathbb{Z}^d} \sum_{u \in \mathbb{N}_{Q_s}^d} I(x - u, x)^2 = \sum_{x \in \mathbb{Z}^d} \sum_{u \in \mathbb{N}_{Q_s}^d} I(u, x)^2 = \sum_{u \in \mathbb{N}_{Q_s}^d} \|I(u, x)\|_{\ell^2(x)}^2$$

where in the second equality periodicity has been used. Next by Proposition 4.1 we obtain

$$\begin{aligned} \sum_{u \in \mathbb{N}_{Q_s}^d} \|I(u, x)\|_{\ell^2(x)}^2 &\lesssim \sum_{u \in \mathbb{N}_{Q_s}^d} \|J(u, x)\|_{\ell^2(x)}^2 = \sum_{u \in \mathbb{N}_{Q_s}^d} \|J(x, x + u)\|_{\ell^2(x)}^2 \\ &= \sum_{u \in \mathbb{N}_{Q_s}^d} \int_{\mathbb{T}^d} \left| \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} G(a/q) e^{2\pi i(a/q) \cdot u} \eta_s(\xi - a/q) \hat{f}(\xi) \right|^2 d\xi \\ &\lesssim (s+1)^{-2\delta l} Q_s^d \cdot \|f\|_{\ell^2}^2. \end{aligned}$$

In the last step we have used (6.4) and the disjointness of supports of  $\eta_s(\cdot - a/q)$  while  $a/q$  varies over  $\mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$ . Therefore,

$$\left\| \sup_{n \geq 2^{\kappa_s}} |\mathcal{F}^{-1}(\nu_{2^n}^s \hat{f})| \right\|_{\ell^2} \lesssim (s+1)^{-\delta l} \|f\|_{\ell^2}$$

which together with (6.23) concludes the proof.  $\square$

**6.4.  $\ell^p(\mathbb{Z}^d)$  estimates for (6.21).** For  $s \in \mathbb{N}_0$  let

$$\kappa_s = 20d(\lfloor (s+1)^{1/10} \rfloor + 1)$$

and

$$Q_s = (\lfloor e^{(s+1)^{1/10}} \rfloor)!$$

be as in the proof of Theorem 6.1. We show that for every  $p \in (1, \infty)$  there is a constant  $C_p > 0$  such that for every  $s \in \mathbb{N}_0$  and all  $(f_t : t \in \mathbb{N}) \in \ell^p(\ell^2(\mathbb{Z}^d))$

$$(6.24) \quad \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} |\mathcal{F}^{-1}(\nu_{2^n}^s \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p s \log(s+2) \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

Then interpolation (6.24) with (6.22) will immediately imply (6.21). The proof of (6.24) will consist of two parts. We shall bound separately the supremum when  $0 \leq n \leq 2^{\kappa_s}$  and when  $n \geq 2^{\kappa_s}$ , see Theorem 6.2 and Theorem 6.4 respectively.

**Theorem 6.2.** *Let  $p \in (1, \infty)$  then there is a constant  $C_p > 0$  such that for any  $s \in \mathbb{N}_0$  and  $(f_t : t \in \mathbb{N}) \in \ell^p(\ell^2(\mathbb{Z}^d))$  we have*

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{0 \leq n \leq 2^{\kappa_s}} |\mathcal{F}^{-1}(\nu_{2^n}^s \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p s \log(s+2) \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

*Proof.* We set  $J = \lfloor e^{(s+1)^{1/2}} \rfloor$  and define

$$\mu_J(\xi) = J^{-k} \sum_{y \in \mathbb{N}_J^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)}$$

where  $\mathbb{N}_J^k = \{1, 2, \dots, J\}^k$ . We see that  $\mu_J$  is the multiplier corresponding to the averaging operator, i.e.  $M_J f = \mathcal{F}^{-1}(\mu_J \hat{f})$ . Thus, we easily note that for each  $r \in [1, \infty]$  we have

$$\|\mathcal{F}^{-1}(\mu_J \hat{f})\|_{\ell^r} \leq \|f\|_{\ell^r}$$

since  $\|K_J\|_{\ell^1} \leq 1$ , where  $K_J$  is the kernel of the operator  $M_J$ , see (6.1). Moreover, if  $\xi \in \mathbb{T}^d$  is such that  $|\xi_\gamma - a_\gamma/q| \leq 2^{-s(|\gamma|-\chi)}$  for every  $\gamma \in \Gamma$  with some  $1 \leq q \leq e^{(s+1)^{1/10}}$  and  $a \in A_q$ , then

$$\mu_J(\xi) = G(a/q) \Phi_J(\xi - a/q) + \mathcal{O}(e^{-\frac{1}{2}(s+1)^{1/2}}).$$

Indeed, by Proposition 6.1 with  $L_1 = 2^s$ ,  $L_2 = 2^{s\chi}$ ,  $L_3 = e^{(s+1)^{1/10}}$  and  $N = J$  we see that the error term is dominated by

$$\begin{aligned} L_3 J^{-1} + L_2 L_3 J^{-1} \sum_{\gamma \in \Gamma} (J/L_1)^{|\gamma|} &\lesssim e^{(s+1)^{1/10} - (s+1)^{1/2}} + 2^{s\chi} e^{(s+1)^{1/10} - (s+1)^{1/2}} (e^{(s+1)^{1/2}} \cdot 2^{-s}) \\ &\lesssim e^{-\frac{1}{2}(s+1)^{1/2}}. \end{aligned}$$

Therefore,

$$(6.25) \quad |\mu_J(\xi) - G(a/q)| \lesssim |G(a/q)(\Phi_J(\xi - a/q) - 1)| + e^{-\frac{1}{2}(s+1)^{1/2}} \lesssim e^{-\frac{1}{2}(s+1)^{1/2}}$$

since

$$|\Phi_J(\xi - a/q) - 1| \lesssim |J^A(\xi - a/q)| \lesssim e^{(s+1)^{1/2}} 2^{-s(1-\chi)}.$$

Let us define the multipliers

$$\Pi_{2^n}^s(\xi) = \sum_{a/q \in \mathcal{W}_{(s+1)^l} \setminus \mathcal{W}_{s^l}} \Phi_{2^n}(\xi - a/q) \eta(2^{s(A-\chi I)}(\xi - a/q))$$

and observe that by (6.25) we have

$$(6.26) \quad \nu_{2^n}^s(\xi) - \mu_J(\xi) \Pi_{2^n}^s(\xi) = \mathcal{O}(e^{-\frac{1}{2}(s+1)^{1/2}}).$$

Hence, by (6.26) and Plancherel's theorem we have

$$(6.27) \quad \|\mathcal{F}^{-1}((\nu_{2^n}^s - \mu_J \Pi_{2^n}^s) \hat{f})\|_{\ell^2} \lesssim e^{-\frac{1}{2}(s+1)^{1/2}} \|f\|_{\ell^2}$$

moreover, for every  $p \in (1, \infty)$  we have a trivial bound

$$(6.28) \quad \|\mathcal{F}^{-1}((\nu_{2^n}^s - \mu_J \Pi_{2^n}^s) \hat{f})\|_{\ell^p} \lesssim |\mathcal{W}_{(s+1)^l}| \cdot \|f\|_{\ell^p} \lesssim e^{(d+1)(s+1)^{1/10}} \|f\|_{\ell^p}.$$

Interpolating now (6.27) with (6.28) one has for some  $c_p > 0$  that

$$(6.29) \quad \|\mathcal{F}^{-1}((\nu_{2^n}^s - \mu_J \Pi_{2^n}^s) \hat{f})\|_{\ell^p} \lesssim e^{-c_p(s+1)^{1/2}} \|f\|_{\ell^p}.$$

Thus in view of (4.6) and (6.29) we obtain

$$\begin{aligned} \left\| \left( \sum_{t \in \mathbb{N}} \sup_{0 \leq n \leq 2^{\kappa_s}} |\mathcal{F}^{-1}((\nu_{2^n}^s - \mu_J \Pi_{2^n}^s) \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} &\lesssim \sum_{n=0}^{2^{\kappa_s}} \left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{F}^{-1}((\nu_{2^n}^s - \mu_J \Pi_{2^n}^s) \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \\ &\lesssim \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p} \end{aligned}$$

since  $2^{\kappa_s} e^{-c_p(s+1)^{1/2}} \lesssim 1$ . The proof of Theorem 6.2 will be completed if we show

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{0 \leq n \leq 2^{\kappa_s}} |\mathcal{F}^{-1}(\Pi_{2^n}^s \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim \kappa_s \log(s+2) \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

Appealing to inequality (2.3) we see that

$$\begin{aligned} \left\| \left( \sum_{t \in \mathbb{N}} \sup_{0 \leq n \leq 2^{\kappa_s}} |\mathcal{F}^{-1}(\Pi_{2^n}^s \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} &\lesssim \left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{F}^{-1}(\Pi_1^s \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \\ &\quad + \sum_{i=0}^{\kappa_s} \left\| \left( \sum_{t \in \mathbb{N}} \sum_{j=0}^{2^{\kappa_s-i}-1} \left| \sum_{m \in I_j^i} \mathcal{F}^{-1}((\Pi_{2^{m+1}}^s - \Pi_{2^m}^s) \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} \end{aligned}$$

where  $I_j^i = [j2^i, (j+1)2^i)$ . In view of Lemma 4.2 it suffices to show that for every  $i \in \{0, 1, \dots, \kappa_s\}$  and  $\omega \in [0, 1]$  we have

$$(6.30) \quad \left\| \sum_{j=0}^{2^{\kappa_s-i}-1} \sum_{m \in I_j^i} \varepsilon_j(\omega) \mathcal{F}^{-1}((\Pi_{2^{m+1}}^s - \Pi_{2^m}^s) \hat{f}) \right\|_{\ell^p} \lesssim \log(s+2) \|f\|_{\ell^p}$$

for any sequence  $\varepsilon = (\varepsilon_j(\omega) : 0 \leq j < 2^{\kappa_s-i})$  with  $\varepsilon_j(\omega) \in \{-1, 1\}$ . Let us consider the operator

$$\mathcal{T}_\varepsilon f = \sum_{a/q \in \mathcal{U}_{(s+1)^t} \setminus \mathcal{U}_s^t} \mathcal{F}^{-1}(\Theta(\cdot - a/q) \eta_s(\cdot - a/q) \hat{f})$$

with

$$\Theta = \sum_{j=0}^{2^{\kappa_s-i}-1} \varepsilon_j(\omega) \sum_{m \in I_j^i} (\Phi_{2^{m+1}} - \Phi_{2^m}).$$

We notice that the multiplier  $\Theta$  corresponds to a continuous singular Radon transform. Thus  $\Theta$  defines a bounded operator on  $L^r(\mathbb{R}^d)$  for any  $r \in (1, \infty)$  with the bound independent of the sequence  $(\varepsilon_j(\omega) : 0 \leq j \leq 2^{\kappa_s-i})$  (see [15, Section 11] or [8]). Hence, by Theorem 5.1

$$\|\mathcal{T}_\varepsilon f\|_{\ell^p} \lesssim \log(s+2) \|f\|_{\ell^p}$$

and consequently we obtain (6.30) and the proof of Theorem 6.2 is completed.  $\square$

For each  $N \in \mathbb{N}$  and  $s \in \mathbb{N}_0$  we define multipliers

$$\Omega_N^s(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^t} \setminus \mathcal{U}_s^t} G(a/q) \Theta_N(\xi - a/q) \varrho_s(\xi - a/q)$$

where  $\varrho_s(\xi) = \eta(Q_{s+1}^{3dA} \xi)$  and  $(\Theta_N : N \in \mathbb{N})$  is the sequence of multipliers on  $\mathbb{R}^d$  obeying (4.1).

**Theorem 6.3.** *Let  $p \in (1, \infty)$  then there exists  $C_p > 0$  such that for any  $s \in \mathbb{N}_0$  and  $(f_t : t \in \mathbb{N}) \in \ell^p(\ell^2(\mathbb{Z}^d))$  we have*

$$\left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{N}(\mathcal{F}^{-1}(\Omega_N^s \hat{f}_t) : N \in \mathbb{N})|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \mathbf{B}_p \log(s+2) \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

*Proof.* Let us observe that

$$\mathcal{F}^{-1}(\Theta_N(\cdot - a/q) \varrho_s(\cdot - a/q) \hat{f})(Q_s x + m) = \mathcal{F}^{-1}(\Theta_N \varrho_s \hat{f}(\cdot + a/q))(Q_s x + m) e^{-2\pi i(a/q) \cdot m}.$$

Therefore,

$$\begin{aligned} &\left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{N}(\mathcal{F}^{-1}(\Omega_N^s \hat{f}_t) : N \in \mathbb{N})|^2 \right)^{1/2} \right\|_{\ell^p}^p \\ &= \sum_{m \in \mathbb{N}_{Q_s}^d} \left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{N}(\mathcal{F}^{-1}(\Theta_N \varrho_s F_t(\cdot; m))(Q_s x + m) : N \in \mathbb{N})|^2 \right)^{1/2} \right\|_{\ell^p(x)}^p \end{aligned}$$

where

$$(6.31) \quad F_t(\xi; m) = \sum_{a/q \in \mathcal{U}_{(s+1)^t} \setminus \mathcal{U}_s^t} G(a/q) \hat{f}_t(\xi + a/q) e^{-2\pi i(a/q) \cdot m}.$$

Now, by Proposition 4.2 and (6.31) we get

$$\begin{aligned} & \sum_{m \in \mathbb{N}_{Q_s}^d} \left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{N}(\mathcal{F}^{-1}(\Theta_N \varrho_s F_t(\cdot; m))(Q_s x + m)) : N \in \mathbb{N}|^2 \right)^{1/2} \right\|_{\ell^p(x)}^p \\ & \leq C_p^p \mathbf{B}_p^p \sum_{m \in \mathbb{N}_{Q_s}^d} \left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{F}^{-1}(\varrho_s F_t(\cdot; m))(Q_s x + m)|^2 \right)^{1/2} \right\|_{\ell^p(x)}^p \\ & = C_p^p \mathbf{B}_p^p \left\| \left( \sum_{t \in \mathbb{N}} \left| \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s l} G(a/q) \mathcal{F}^{-1}(\varrho_s(\cdot - a/q) \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p}^p. \end{aligned}$$

In view of (4.6) it suffices to prove

$$(6.32) \quad \|\mathcal{F}^{-1}(\tilde{\Pi}_s^G \hat{f})\|_{\ell^p} \lesssim \log(s+2) \|f\|_{\ell^p}$$

where

$$\tilde{\Pi}_s^G(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s l} G(a/q) \varrho_s(\xi - a/q).$$

Observe that arguing in a similar way as in the proof of Theorem 6.2 we obtain by (6.25) that

$$(6.33) \quad |\tilde{\Pi}_s^G(\xi) - \mu_J(\xi) \tilde{\Pi}_s(\xi)| \lesssim e^{-\frac{1}{2}(s+1)^{1/2}}$$

where

$$\tilde{\Pi}_s(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s l} \varrho_s(\xi - a/q).$$

Therefore, (6.33) combined with Plancherel's theorem yields

$$(6.34) \quad \|\mathcal{F}^{-1}((\tilde{\Pi}_s^G - \mu_J \tilde{\Pi}_s) \hat{f})\|_{\ell^2} \lesssim e^{-\frac{1}{2}(s+1)^{1/2}} \|f\|_{\ell^2}.$$

We can conclude by interpolation with (6.34) that

$$\|\mathcal{F}^{-1}((\tilde{\Pi}_s^G - \mu_J \tilde{\Pi}_s) \hat{f})\|_{\ell^p} \lesssim \|f\|_{\ell^p}.$$

since by Theorem 5.1 we have

$$\|\mathcal{F}^{-1}(\tilde{\Pi}_s \hat{f})\|_{\ell^p} \lesssim \log(s+2) \|f\|_{\ell^p}$$

and by the trivial bound

$$\|\mathcal{F}^{-1}(\tilde{\Pi}_s^G \hat{f})\|_{\ell^p} \lesssim |\mathcal{U}_{(s+1)^l}| \cdot \|f\|_{\ell^p} \lesssim e^{(d+1)(s+1)^{1/10}} \|f\|_{\ell^p}.$$

This establishes the bound in (6.32) and the proof of Theorem 6.3 is finished.  $\square$

**Theorem 6.4.** *Let  $p \in (1, \infty)$  then there is a constant  $C_p > 0$  such that for any  $s \in \mathbb{N}_0$  and  $(f_t : t \in \mathbb{N}) \in \ell^p(\ell^2(\mathbb{Z}^d))$  we have*

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \geq 2^{\kappa_s}} |\mathcal{F}^{-1}(\nu_{2^n}^s \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \log(s+2) \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

*Proof.* Theorem A.1 guarantees that the sequence  $(\Phi_{2^n} : n \in \mathbb{N}_0)$  satisfies (4.1). Thus in view of Theorem 6.3 with  $\Theta_N = \Phi_N$  and (4.6) it only suffices to prove that for any  $n \geq 2^{\kappa_s}$  we have

$$(6.35) \quad \|\mathcal{F}^{-1}((\nu_{2^n}^s - \Omega_{2^n}^s) \hat{f})\|_{\ell^p} \lesssim 2^{-c_p^1 n} e^{c_p^2 (s+1)^{1/10}} \|f\|_{\ell^p}$$

for some  $c_p^1, c_p^2 > 0$ . Obviously we have

$$(6.36) \quad \|\mathcal{F}^{-1}((\nu_{2^n}^s - \Omega_{2^n}^s) \hat{f})\|_{\ell^p} \lesssim |\mathcal{U}_{(s+1)^l}| \cdot \|f\|_{\ell^p} \lesssim e^{(d+1)(s+1)^{1/10}} \|f\|_{\ell^p}.$$

Next, we observe that  $\varrho_s(\xi - a/q) - \eta_s(\xi - a/q) \neq 0$  implies that  $|\xi_\gamma - a_\gamma/q| \geq (16d)^{-1} Q_{s+1}^{-3d|\gamma|}$  for some  $\gamma \in \Gamma$ . Therefore, for  $n \geq 2^{\kappa_s}$  we have

$$2^{n|\gamma|} \cdot |\xi_\gamma - a_\gamma/q| \gtrsim 2^{n|\gamma|} Q_{s+1}^{-3d|\gamma|} \gtrsim 2^{n/2},$$

since

$$2^{n/2} Q_{s+1}^{-3d} \geq 2^{2^{\kappa_s-1}} e^{-3d(s+1)^{1/10}} e^{(s+1)^{1/10}} \geq e^{(s+1)^{1/10}}$$

for sufficiently large  $s \in \mathbb{N}_0$ . Thus using (6.2), we obtain

$$|\Phi_{2^n}(\xi - a/q)| \lesssim 2^{-n/(2d)}.$$

Hence, by (6.4)

$$\left| \sum_{a/q \in \mathcal{W}_{(s+1)^l} \setminus \mathcal{W}_s^l} G(a/q) \Phi_{2^n}(\xi - a/q) (\eta_s(\xi - a/q) - \varrho_s(\xi - a/q)) \right| \leq C(s+1)^{-\delta l} 2^{-n/(2d)}.$$

Thus, by Plancherel's theorem we obtain

$$(6.37) \quad \left\| \mathcal{F}^{-1}((\nu_{2^n}^s - \Omega_{2^n}^s) \hat{f}) \right\|_{\ell^2} \lesssim 2^{-n/(2d)} (s+1)^{-\delta l} \|f\|_{\ell^2}.$$

Interpolating now (6.37) with (6.36) we obtain (6.35) and this completes the proof of Theorem 6.4.  $\square$

## 7. VECTOR-VALUED MAXIMAL ESTIMATES FOR TRUNCATED SINGULAR INTEGRAL OPERATORS

The purpose of this section is to prove Theorem C however we begin with some useful reductions. In view of Lemma 2.1, to prove Theorem C it is enough to consider  $T_N^{\mathcal{Q}}$  where  $\mathcal{Q}$  is the canonical polynomial as in Section 2.

Given a kernel  $K$  satisfying (1.1) and (1.2) there are functions  $(K_j : j \geq 0)$  and a constant  $C > 0$  such that for  $x \neq 0$

$$K(x) = \sum_{j=0}^{\infty} K_j(x),$$

where for each  $j \geq 0$  the kernel  $K_j$  is supported inside  $2^{j-2} \leq |x| \leq 2^j$ , satisfies

$$|x|^k |K_j(x)| + |x|^{k+1} |\nabla K_j(x)| \leq C$$

for all  $x \in \mathbb{R}^k$  such that  $|x| \geq 1$ , and has integral 0, provided  $j \geq 1$  (for the proof see [15, Chapter 6, §4.5, Chapter 13, §5.3]).

Next, we define a sequence  $(m_j : j \geq 0)$  of functions on  $\mathbb{T}^d$  by

$$m_j(\xi) = \sum_{y \in \mathbb{Z}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_j(y).$$

Then for any finitely supported function  $f$  on  $\mathbb{Z}^d$

$$\mathcal{F}^{-1}(m_j \hat{f})(x) = \sum_{y \in \mathbb{Z}^k} f(x - \mathcal{Q}(y)) K_j(y).$$

For  $j \geq 0$  we set

$$\Phi_j(\xi) = \int_{\mathbb{R}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_j(y) dy.$$

Using multi-dimensional version of van der Corput's lemma (see [17, Proposition 2.1]) we obtain

$$(7.1) \quad |\Phi_j(\xi)| \lesssim \min \{1, |2^j A \xi|_{\infty}\}^{-1/d}$$

where  $A$  is the  $d \times d$  diagonal matrix as in Section 2. Moreover, if  $j \geq 1$  we have

$$(7.2) \quad |\Phi_j(\xi)| = \left| \Phi_j(\xi) - \int_{\mathbb{R}^k} K_j(y) dy \right| \lesssim \min \{1, |2^j A \xi|_{\infty}\}.$$

Finally, let  $\Psi_n = \sum_{j=0}^n \Phi_j$ .

Now instead of Theorem C we prove the following dyadic version.

**Theorem 7.1.** *For every  $1 < p < \infty$  there is  $C_p > 0$  such that for all  $(f_t : t \in \mathbb{N}) \in \ell^p(\ell^2(\mathbb{Z}^d))$*

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} \left| \sum_{j=0}^n \mathcal{F}^{-1}(m_j \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

It is easy to see that Theorem 7.1 combined with Theorem B implies Theorem C, since for any  $f \geq 0$  we have the following pointwise bound

$$\sup_{N \in \mathbb{N}} |T_N f(x)| \lesssim \sup_{n \in \mathbb{N}_0} \left| \sum_{j=0}^n \mathcal{F}^{-1}(m_j \hat{f})(x) \right| + \sup_{N \in \mathbb{N}} |M_N f(x)|.$$

The strategy of the proof of Theorem 7.1 is much the same as for the proof of the result for averaging operators from Section 6. However, there are some subtle changes which could cause some confusions. Therefore, for the convenience of the reader we provide all details.

**Proposition 7.1.** *There is a constant  $C > 0$  such that for every  $j \in \mathbb{N}$  and for every  $\xi \in [0, 1)^d$  satisfying*

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \leq L_1^{-|\gamma|} L_2$$

for all  $\gamma \in \Gamma$ , where  $1 \leq q \leq L_3 \leq 2^{j/2}$ ,  $a \in A_q$ ,  $L_1 \geq 2^j$  and  $L_2 \geq 1$  we have

$$|m_j(\xi) - G(a/q)\Phi_j(\xi - a/q)| \leq C \left( L_3 2^{-j} + L_2 L_3 2^{-j} \sum_{\gamma \in \Gamma} (2^j/L_1)^{|\gamma|} \right) \leq C L_2 L_3 2^{-j}.$$

*Proof.* Let  $\theta = \xi - a/q$ . For any  $r \in \mathbb{N}_q^k$ , if  $y \equiv r \pmod{q}$  then for each  $\gamma \in \Gamma$

$$\xi_\gamma y^\gamma \equiv \theta_\gamma y^\gamma + (a_\gamma/q) r^\gamma \pmod{1},$$

thus

$$\xi \cdot \mathcal{Q}(y) \equiv \theta \cdot \mathcal{Q}(y) + (a/q) \cdot \mathcal{Q}(r) \pmod{1}.$$

Therefore,

$$\sum_{y \in \mathbb{Z}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_j(y) = \sum_{r \in \mathbb{N}_q^k} e^{2\pi i (a/q) \cdot \mathcal{Q}(r)} \sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy+r)} K_j(qy+r).$$

If  $2^{j-2} \leq |qy+r|, |qy| \leq 2^j$  then

$$|\theta \cdot \mathcal{Q}(qy+r) - \theta \cdot \mathcal{Q}(qy)| \lesssim |r| \sum_{\gamma \in \Gamma} |\theta_\gamma| \cdot 2^{j(|\gamma|-1)} \lesssim q \sum_{\gamma \in \Gamma} L_1^{-|\gamma|} L_2 2^{j(|\gamma|-1)} \lesssim L_2 L_3 2^{-j} \sum_{\gamma \in \Gamma} (2^j/L_1)^{|\gamma|}.$$

and

$$|K_j(qy+r) - K_j(qy)| \lesssim 2^{-j(k+1)} L_3.$$

Thus

$$\sum_{y \in \mathbb{Z}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_j(y) = G(a/q) \cdot q^k \sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K_j(qy) + \mathcal{O} \left( L_3 2^{-j} + L_2 L_3 2^{-j} \sum_{\gamma \in \Gamma} (2^j/L_1)^{|\gamma|} \right).$$

Again by the mean value theorem one can replace the sum on the right-hand side by the integral. Indeed, we obtain

$$\begin{aligned} & \left| \sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K_j(qy) - \int_{\mathbb{R}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qt)} K_j(qt) dt \right| \\ &= \left| \sum_{y \in \mathbb{Z}^k} \int_{[0,1)^k} (e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K_j(qy) - e^{2\pi i \theta \cdot \mathcal{Q}(q(y+t))} K_j(q(y+t))) dt \right| \\ &= \mathcal{O} \left( q^{-k} L_3 2^{-j} + q^{-k} L_2 2^{-j} \sum_{\gamma \in \Gamma} (2^j/L_1)^{|\gamma|} \right). \end{aligned}$$

This completes the proof of Proposition 7.1.  $\square$

As in Section 6 let  $\chi > 0$  and  $l \in \mathbb{N}$  be the numbers whose precise values will be chosen later. For every  $j \in \mathbb{N}_0$  we define the multipliers

$$(7.3) \quad \Xi_j(\xi) = \sum_{a/q \in \mathcal{U}_{j,l}} \eta(2^{j(A-\chi I)}(\xi - a/q))$$

with  $\mathcal{U}_{n,l}$  defined as in Section 5. Theorem 5.1 guarantees that for every  $p \in (1, \infty)$

$$(7.4) \quad \|\mathcal{F}^{-1}(\Xi_j \hat{f})\|_{\ell^p} \lesssim \log(j+2) \|f\|_{\ell^p}.$$

The implicit constant in (7.4) depends on the parameter  $\rho > 0$ , which was fixed, see Section 5. However, from now on we will assume that  $\rho > 0$  and the integer  $l \geq 10$  are related by the equation

$$(7.5) \quad 10\rho l = 1.$$

Observe, with the aid of (7.3), that

$$\begin{aligned} (7.6) \quad & \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} \left| \sum_{j=0}^n \mathcal{F}^{-1}(m_j \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} \leq \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} \left| \sum_{j=0}^n \mathcal{F}^{-1}(m_j \Xi_j \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} \\ & + \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} \left| \sum_{j=0}^n \mathcal{F}^{-1}(m_j (1 - \Xi_j) \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p}. \end{aligned}$$



**7.1. The estimate for the second norm in (7.6).** We see that

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} \left| \sum_{j=0}^n \mathcal{F}^{-1}(m_j(1 - \Xi_j) \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} \leq \sum_{j \in \mathbb{N}_0} \left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{F}^{-1}(m_j(1 - \Xi_j) \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p}.$$

Therefore, in a similar way as in the subsection concerning the estimates for the second norm in (6.7) it is possible to show, using Theorem 3.1, that

$$\left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{F}^{-1}(m_j(1 - \Xi_j) \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \leq (j+1)^{-2} \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

**7.2. The estimate for the first norm in (6.7).** Note that for any  $\xi \in \mathbb{T}^d$  so that

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \leq (8d)^{-1} 2^{-j(|\gamma| - \chi)}$$

for every  $\gamma \in \Gamma$  with  $1 \leq q \leq e^{j^{1/10}}$  we have

$$(7.7) \quad m_j(\xi) = G(a/q) \Phi_j(\xi - a/q) + q^{-\delta} E_j(\xi)$$

where

$$(7.8) \quad |E_j(\xi)| \lesssim 2^{-j/2}.$$

These two properties (7.7) and (7.8) follow from Proposition 7.1 with  $L_1 = 2^j$ ,  $L_2 = 2^{\chi j}$  and  $L_3 = e^{j^{1/10}}$ , since

$$|E_j(\xi)| \lesssim q^\delta L_2 L_3 2^{-j} \lesssim (e^{-j((1-\chi) \log 2 - 2j^{-9/10})}) \lesssim 2^{-j/2}$$

which holds for sufficiently large  $j \in \mathbb{N}$ , when  $\chi > 0$  is sufficiently small. Let us introduce for every  $j \in \mathbb{N}_0$  new multipliers

$$\nu_j(\xi) = \sum_{a/q \in \mathcal{U}_{j,l}} G(a/q) \Phi_j(\xi - a/q) \eta(2^{j(A-\chi I)}(\xi - a/q)).$$

In a similar way as in the previous section (see (6.16)) one can conclude that for some  $c_p > 0$

$$(7.9) \quad \left\| \mathcal{F}^{-1}((m_j \Xi_j - \nu_j) \hat{f}) \right\|_{\ell^p} \lesssim 2^{-c_p j} \|f\|_{\ell^p}.$$

For every  $j, s \in \mathbb{N}_0$  define multipliers

$$\nu_j^s(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)l} \setminus \mathcal{U}_{s,l}} G(a/q) \Phi_j(\xi - a/q) \eta(2^{s(A-\chi I)}(\xi - a/q)).$$

Arguing as in the proof of estimate (6.20) we see that for some  $c_p > 0$

$$(7.10) \quad \left\| \mathcal{F}^{-1}((\nu_j - \sum_{0 \leq s < j} \nu_j^s) \hat{f}) \right\|_{\ell^p} \lesssim 2^{-c_p j} \|f\|_{\ell^p}.$$

Therefore in view of (4.6) and (7.9) and (7.10) it suffices to prove that for every  $s \in \mathbb{N}_0$  we have

$$(7.11) \quad \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} \left| \sum_{j=0}^n \mathcal{F}^{-1}(\nu_j^s \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim (s+1)^{-2} \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

**7.3.  $\ell^2(\mathbb{Z}^d)$  estimates for (7.11).** We shall prove

$$(7.12) \quad \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} \left| \sum_{j=0}^n \mathcal{F}^{-1}(\nu_j^s \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^2} \lesssim (s+1)^{-\delta l + 1} \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^2}.$$

For the proof of (7.12) it suffices to show the following.

**Theorem 7.2.** *There is  $C > 0$  such that for any  $s \in \mathbb{N}_0$  and  $f \in \ell^2(\mathbb{Z}^d)$*

$$\left\| \sup_{n \in \mathbb{N}_0} \left| \sum_{j=0}^n \mathcal{F}^{-1}(\nu_j^s \hat{f}) \right| \right\|_{\ell^2} \leq C(s+1)^{-\delta l + 1} \|f\|_{\ell^2}$$

with  $l \in \mathbb{N}$  defined in (7.5) and  $\delta > 0$  as in (6.4).

*Proof.* We shall proceed analogously to the proof of Theorem 6.1. For this purpose for  $s \in \mathbb{N}_0$  we set  $\kappa_s = 20d(\lfloor (s+1)^{1/10} \rfloor + 1)$  and  $Q_s = (\lfloor e^{(s+1)^{1/10}} \rfloor)!$ . Firstly, we estimate the supremum over  $0 \leq n \leq 2^{\kappa_s}$ . By Lemma 2.2 we have

$$\left\| \sup_{0 \leq n \leq 2^{\kappa_s}} \left| \sum_{j=0}^n \mathcal{F}^{-1}(\nu_j^s \hat{f}) \right| \right\|_{\ell^2} \lesssim \|\mathcal{F}^{-1}(\nu_0^s \hat{f})\|_{\ell^2} + \sum_{i=0}^{\kappa_s} \left( \sum_{j=0}^{2^{\kappa_s-i}-1} \left\| \sum_{n \in I_j^i} \mathcal{F}^{-1}(\nu_n^s \hat{f}) \right\|_{\ell^2}^2 \right)^{1/2}$$

where  $I_j^i = (j2^i, (j+1)2^i]$ . For any  $i \in \{0, \dots, \kappa_s\}$ , by Plancherel's theorem we get

$$\begin{aligned} \sum_{j=0}^{2^{\kappa_s-i}-1} \left\| \sum_{n \in I_j^i} \mathcal{F}^{-1}(\nu_n^s \hat{f}) \right\|_{\ell^2}^2 &\leq \sum_{j=0}^{2^{\kappa_s-i}-1} \sum_{n, n' \in I_j^i} \int_{\mathbb{T}^d} |\nu_n^s(\xi)| \cdot |\nu_{n'}^s(\xi)| \cdot |\hat{f}(\xi)|^2 d\xi \\ &\leq \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} |G(a/q)|^2 \sum_{j=0}^{2^{\kappa_s-i}-1} \sum_{n, n' \in I_j^i} \int_{\mathbb{T}^d} |\Phi_n(\xi - a/q)| \cdot |\Phi_{n'}(\xi - a/q)| \cdot \eta_s(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi. \end{aligned}$$

where  $\eta_s(\xi) = \eta(2^{s(A-\chi I)}\xi)$ , since the supports are effectively disjoint. Using (7.1) and (7.2) we conclude

$$\sum_{n \in \mathbb{Z}} |\Phi_n(\xi - a/q)| \lesssim \sum_{n \in \mathbb{Z}} \min \{ |2^{nA}\xi|_{\infty}, |2^{nA}\xi|_{\infty}^{-1/d} \} \lesssim 1.$$

Therefore, by (6.4) we may estimate

$$\sum_{j=0}^{2^{\kappa_s-i}-1} \left\| \sum_{n \in I_j^i} \mathcal{F}^{-1}(\nu_n^s \hat{f}) \right\|_{\ell^2}^2 \lesssim (s+1)^{-2\delta l} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} \int_{\mathbb{T}^d} \eta_s(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \lesssim (s+1)^{-2\delta l} \|f\|_{\ell^2}^2.$$

where in the last step we have used disjointness of supports of  $\eta_s(\cdot - a/q)$  while  $a/q$  varies over  $\mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$  and the fact that if  $a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$  then  $q \geq s^l$ . We have just proven

$$(7.13) \quad \left\| \sup_{0 \leq n \leq 2^{\kappa_s}} \left| \sum_{j=0}^n \mathcal{F}^{-1}(\nu_j^s \hat{f}) \right| \right\|_{\ell^2} \lesssim \kappa_s (s+1)^{-\delta l} \|f\|_{\ell^2} \lesssim (s+1)^{-\delta l+1} \|f\|_{\ell^2}.$$

Next, we consider the case when the supremum is taken over  $n \geq 2^{\kappa_s}$ . For any  $x, y \in \mathbb{Z}^d$  we define

$$I(x, y) = \sup_{n \geq 2^{\kappa_s}} \left| \sum_{j=2^{\kappa_s}}^n \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} G(a/q) e^{-2\pi i(a/q) \cdot x} \mathcal{F}^{-1}(\Phi_j \eta_s \hat{f}(\cdot + a/q))(y) \right|$$

and

$$J(x, y) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} G(a/q) e^{-2\pi i(a/q) \cdot x} \mathcal{F}^{-1}(\eta_s \hat{f}(\cdot + a/q))(y).$$

By Plancherel's theorem, for any  $u \in \mathbb{N}_{Q_s}^d$  and  $a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$  we have

$$\begin{aligned} &\left\| \mathcal{F}^{-1}(\Phi_j \eta_s \hat{f}(\cdot + a/q))(x+u) - \mathcal{F}^{-1}(\Phi_j \eta_s \hat{f}(\cdot + a/q))(x) \right\|_{\ell^2(x)} \\ &= \left\| (1 - e^{-2\pi i \xi \cdot u}) \Phi_j(\xi) \eta_s(\xi) \hat{f}(\xi + a/q) \right\|_{L^2(d\xi)} \\ &\lesssim 2^{-j/d} \cdot |u| \cdot \left\| \eta_s(\cdot - a/q) \hat{f} \right\|_{L^2} \end{aligned}$$

since, by (7.1),

$$\sup_{\xi \in \mathbb{T}^d} |\xi| \cdot |\Phi_j(\xi)| \lesssim \sup_{\xi \in \mathbb{T}^d} |\xi| \cdot |2^{jA}\xi|^{-1/d} \leq 2^{-j/d}.$$

Therefore,

$$\left| \|I(x, x+u)\|_{\ell^2(x)} - \|I(x, x)\|_{\ell^2(x)} \right| \leq |u| \sum_{j=2^{\kappa_s}}^{\infty} 2^{-j/d} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} \left\| \eta_s(\cdot - a/q) \hat{f} \right\|_{\ell^2}$$

because the set  $\mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$  contains at most  $e^{(d+1)(s+1)^{1/10}}$  elements and

$$2^{\kappa_s} (\log 2)/d - (s+1)^{1/10} e^{(s+1)^{1/10}} - (d+1)(s+1)^{1/10} \geq s$$

for sufficiently large  $s \geq 0$ . Hence, we obtain

$$\|I(x, x)\|_{\ell^2(x)} \lesssim \|I(x, x+u)\|_{\ell^2(x)} + 2^{-s}\|f\|_{\ell^2}.$$

In particular,

$$\left\| \sup_{n \geq 2^{\kappa_s}} \left| \sum_{j=2^{\kappa_s}}^n \mathcal{F}^{-1}(\nu_j^s \hat{f}) \right| \right\|_{\ell^2}^2 \lesssim \frac{1}{Q_s^d} \sum_{u \in \mathbb{N}_{Q_s}^d} \|I(x, x+u)\|_{\ell^2(x)}^2 + 2^{-2s}\|f\|_{\ell^2}^2.$$

Let us observe that the functions  $x \mapsto I(x, y)$  and  $x \mapsto J(x, y)$  are  $Q_s \mathbb{Z}^d$ -periodic. Next, by double change of variables and periodicity we get

$$\sum_{u \in \mathbb{N}_{Q_s}^d} \|I(x, x+u)\|_{\ell^2(x)}^2 = \sum_{x \in \mathbb{Z}^d} \sum_{u \in \mathbb{N}_{Q_s}^d} I(x-u, x)^2 = \sum_{x \in \mathbb{Z}^d} \sum_{u \in \mathbb{N}_{Q_s}^d} I(u, x)^2 = \sum_{u \in \mathbb{N}_{Q_s}^d} \|I(u, x)\|_{\ell^2(x)}^2.$$

Thus, using Proposition 4.1 and (6.4), we have

$$\begin{aligned} \sum_{u \in \mathbb{N}_{Q_s}^d} \|I(u, x)\|_{\ell^2(x)}^2 &\lesssim \sum_{u \in \mathbb{N}_{Q_s}^d} \|J(u, x)\|_{\ell^2(x)}^2 = \sum_{u \in \mathbb{N}_{Q_s}^d} \|J(x, x+u)\|_{\ell^2(x)}^2 \\ &= \sum_{u \in \mathbb{N}_{Q_s}^d} \int_{\mathbb{T}^d} \left| \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} G(a/q) e^{2\pi i(a/q) \cdot u} \eta_s(\xi - a/q) \hat{f}(\xi) \right|^2 d\xi \\ &\lesssim (s+1)^{-2\delta l} Q_s^d \cdot \|f\|_{\ell^2}^2. \end{aligned}$$

In the last step we have also used the disjointness of supports of  $\eta_s(\cdot - a/q)$  while  $a/q$  varies over  $\mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$ . Therefore,

$$\left\| \sup_{n \geq 2^{\kappa_s}} \left| \sum_{j=2^{\kappa_s}}^n \mathcal{F}^{-1}(\nu_j^s \hat{f}) \right| \right\|_{\ell^2} \lesssim (s+1)^{-\delta l} \|f\|_{\ell^2}$$

which together with (7.13) concludes the proof.  $\square$

**7.4.  $\ell^p(\mathbb{Z}^d)$  estimates for (7.11).** Recall from the proof of Theorem 6.1 or Theorem 7.2, that for  $s \in \mathbb{N}_0$

$$\kappa_s = 20d(\lfloor (s+1)^{1/10} \rfloor + 1)$$

and

$$Q_s = (\lfloor e^{(s+1)^{1/10}} \rfloor!).$$

We show that for every  $p \in (1, \infty)$  there is a constant  $C_p > 0$  such that for any  $s \in \mathbb{N}_0$  and  $(f_t : t \in \mathbb{N}) \in \ell^p(\ell^2(\mathbb{Z}^d))$  we have

$$(7.14) \quad \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} \left| \sum_{j=0}^n \mathcal{F}^{-1}(\nu_j^s \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p s \log(s+2) \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

Then interpolation (7.14) with (7.12) will immediately imply (7.11). The proof of (7.14) will consist of two parts. We shall bound separately the supremum when  $0 \leq n \leq 2^{\kappa_s}$  and when  $n \geq 2^{\kappa_s}$ , see Theorem 7.3 and Theorem 7.4 respectively.

**Theorem 7.3.** *Let  $p \in (1, \infty)$  then there is a constant  $C_p > 0$  such that for any  $s \in \mathbb{N}_0$  and  $(f_t : t \in \mathbb{N}) \in \ell^p(\ell^2(\mathbb{Z}^d))$  we have*

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{0 \leq n \leq 2^{\kappa_s}} \left| \sum_{j=0}^n \mathcal{F}^{-1}(\nu_j^s \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p s \log(s+2) \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

*Proof.* We set  $J = \lfloor e^{(s+1)^{1/2}} \rfloor$  and define

$$\mu_J(\xi) = J^{-k} \sum_{y \in \mathbb{N}_J^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)}$$

where  $\mathbb{N}_J^k = \{1, 2, \dots, J\}^k$ . Recall that  $\mu_J$  is the multiplier corresponding to the averaging operator, i.e.  $M_J f = \mathcal{F}^{-1}(m_J \hat{f})$ . Thus we note that for each  $r \in [1, \infty]$  we have

$$\|\mathcal{F}^{-1}(\mu_J \hat{f})\|_{\ell^r} \leq \|f\|_{\ell^r}.$$

Moreover, if  $\xi \in \mathbb{T}^d$  is such that  $|\xi_\gamma - a_\gamma/q| \leq 2^{-s(|\gamma|-\chi)}$  for every  $\gamma \in \Gamma$  with some  $1 \leq q \leq e^{(s+1)^{1/10}}$  and  $a \in A_q$ , then

$$(7.15) \quad |\mu_J(\xi) - G(a/q)| \lesssim e^{-\frac{1}{2}(s+1)^{1/2}}$$

due to Proposition 6.1. We refer to the proof of Theorem 6.2 for more details. Let us define the multipliers

$$\Pi_j^s(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)t} \setminus \mathcal{U}_{st}} \Phi_j(\xi - a/q) \eta(2^{s(A-\chi I)}(\xi - a/q))$$

and observe that by (7.15) we have

$$\nu_j^s(\xi) - \mu_J(\xi) \Pi_j^s(\xi) = \mathcal{O}(e^{-\frac{1}{2}(s+1)^{1/2}}).$$

Hence, arguing as in the previous section we obtain that for some  $c_p > 0$

$$(7.16) \quad \|\mathcal{F}^{-1}((\nu_j^s - \mu_J \Pi_j^s) \hat{f})\|_{\ell^p} \lesssim e^{-c_p(s+1)^{1/2}} \|f\|_{\ell^p}.$$

Thus in view of (4.6) and (7.16) we obtain

$$\begin{aligned} \left\| \left( \sum_{t \in \mathbb{N}} \sup_{0 \leq n \leq 2^{\kappa_s}} \left| \sum_{j=0}^n \mathcal{F}^{-1}((\nu_j^s - \mu_J \Pi_j^s) \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} &\lesssim \sum_{j=0}^{2^{\kappa_s}} \left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{F}^{-1}((\nu_j^s - \mu_J \Pi_j^s) \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \\ &\lesssim \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p} \end{aligned}$$

since  $2^{\kappa_s} e^{-c_p(s+1)^{1/2}} \lesssim 1$ . The proof of Theorem 7.3 will be completed if we show

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{0 \leq n \leq 2^{\kappa_s}} \left| \sum_{j=0}^n \mathcal{F}^{-1}(\Pi_j^s \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim \kappa_s \log(s+2) \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

Appealing to inequality (2.3) we see that

$$\begin{aligned} \left\| \left( \sum_{t \in \mathbb{N}} \sup_{0 \leq n \leq 2^{\kappa_s}} \left| \sum_{j=0}^n \mathcal{F}^{-1}(\Pi_j^s \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} &\lesssim \left\| \left( \sum_{t \in \mathbb{N}} |\mathcal{F}^{-1}(\Pi_0^s \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \\ &\quad + \sum_{i=0}^{\kappa_s} \left\| \left( \sum_{t \in \mathbb{N}} \sum_{j=0}^{2^{\kappa_s-i}-1} \left| \sum_{m \in I_j^i} \mathcal{F}^{-1}(\Pi_m^s \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} \end{aligned}$$

where  $I_j^i = (j2^i, (j+1)2^i]$ . In view of Lemma 4.2 it suffices to show that for every  $i \in \{0, 1, \dots, \kappa_s\}$  and  $\omega \in [0, 1]$  we have

$$(7.17) \quad \left\| \sum_{j=0}^{2^{\kappa_s-i}-1} \sum_{m \in I_j^i} \varepsilon_j(\omega) \mathcal{F}^{-1}(\Pi_m^s \hat{f}) \right\|_{\ell^p} \lesssim \log(s+2) \|f\|_{\ell^p}.$$

for any sequence  $\varepsilon = (\varepsilon_j(\omega) : 0 \leq j < 2^{\kappa_s-i})$  with  $\varepsilon_j(\omega) \in \{-1, 1\}$ . Let us consider the operator

$$\mathcal{T}_\varepsilon f = \sum_{a/q \in \mathcal{U}_{(s+1)t} \setminus \mathcal{U}_{st}} \mathcal{F}^{-1}(\Theta(\cdot - a/q) \eta_s(\cdot - a/q) \hat{f})$$

with

$$\Theta = \sum_{j=0}^{2^{\kappa_s-i}-1} \varepsilon_j(\omega) \sum_{m \in I_j^i} \Phi_m.$$

We notice that the multiplier  $\Theta$  corresponds to a continuous singular Radon transform. Thus  $\Theta$  defines a bounded operator on  $L^r(\mathbb{R}^d)$  for any  $r \in (1, \infty)$  with the bound independent of the sequence of signs  $(\varepsilon_j(\omega) : 0 \leq j \leq 2^{\kappa_s-i})$  (see [15, Section 11] or [8]). Hence, by Theorem 5.1

$$\|\mathcal{T}_\varepsilon f\|_{\ell^p} \lesssim \log(s+2) \|f\|_{\ell^p}$$

and consequently we obtain (7.17) and the proof of Theorem 6.2 is completed.  $\square$

**Theorem 7.4.** *Let  $p \in (1, \infty)$  then there is a constant  $C_p > 0$  such that for any  $s \in \mathbb{N}_0$  and  $(f_t : t \in \mathbb{N}) \in \ell^p(\ell^2(\mathbb{Z}^d))$  we have*

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \geq 2^{\kappa_s}} \left| \sum_{j=0}^n \mathcal{F}^{-1}(\nu_j^s \hat{f}_t) \right|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \log(s+2) \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

*Proof.* The proof of Theorem A.2 guarantees that the sequence  $(\Psi_n : n \in \mathbb{N}_0)$  with  $\Psi_n = \sum_{j=0}^n \Phi_j$  satisfies (4.1). Therefore Theorem 6.3 with  $\Theta_n = \Psi_n$  yields

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{N}_0} |\mathcal{F}^{-1}(\Omega_n^s \hat{f}_t)|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \mathbf{B}_p \log(s+2) \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{\ell^p}.$$

Thus in view of (4.6) it only suffices to prove that for any  $j \geq 2^{\kappa_s}$  we have

$$(7.18) \quad \left\| \mathcal{F}^{-1}((\nu_j^s - \tilde{\Omega}_j^s) \hat{f}) \right\|_{\ell^p} \lesssim 2^{-c_p^1 j} e^{c_p^2(s+1)^{1/10}} \|f\|_{\ell^p}$$

for some  $c_p^1, c_p^2 > 0$ , where

$$\tilde{\Omega}_j^s(\xi) = \sum_{a/q \in \mathcal{W}_{(s+1)^t} \setminus \mathcal{W}_s^t} G(a/q) \Phi_j(\xi - a/q) \varrho_s(\xi - a/q)$$

and we have  $\Omega_n^s = \sum_{j=0}^n \tilde{\Omega}_j^s$ . Now observe that

$$(7.19) \quad \left\| \mathcal{F}^{-1}((\nu_j^s - \tilde{\Omega}_j^s) \hat{f}) \right\|_{\ell^p} \lesssim |\mathcal{W}_{(s+1)^t}| \cdot \|f\|_{\ell^p} \lesssim e^{(d+1)(s+1)^{1/10}} \|f\|_{\ell^p}.$$

Next, we observe that  $\varrho_s(\xi - a/q) - \eta_s(\xi - a/q) \neq 0$  implies that  $|\xi_\gamma - a_\gamma/q| \geq (16d)^{-1} Q_{s+1}^{-3d|\gamma|}$  for some  $\gamma \in \Gamma$ . Therefore, for  $j \geq 2^{\kappa_s}$  we have

$$2^{j|\gamma|} \cdot |\xi_\gamma - a_\gamma/q| \gtrsim 2^{j|\gamma|} Q_{s+1}^{-3d|\gamma|} \gtrsim 2^{j/2},$$

since

$$2^{j/2} Q_{s+1}^{-3d} \geq 2^{2^{\kappa_s}-1} e^{-3d(s+1)^{1/10}} e^{(s+1)^{1/10}} \geq e^{(s+1)^{1/10}}$$

for sufficiently large  $s \in \mathbb{N}_0$ . Using (7.1), we obtain

$$|\Phi_j(\xi - a/q)| \lesssim 2^{-j/(2d)}.$$

Hence, by (6.4)

$$\left| \sum_{a/q \in \mathcal{W}_{(s+1)^t} \setminus \mathcal{W}_s^t} G(a/q) \Phi_j(\xi - a/q) (\eta_s(\xi - a/q) - \varrho_s(\xi - a/q)) \right| \leq C(s+1)^{-\delta l} 2^{-j/(2d)}.$$

Thus, by Plancherel's theorem we obtain

$$(7.20) \quad \left\| \mathcal{F}^{-1}((\nu_j^s - \tilde{\Omega}_j^s) \hat{f}) \right\|_{\ell^2} \lesssim 2^{-j/(2d)} (s+1)^{-\delta l} \|f\|_{\ell^2}.$$

Interpolating now (7.20) with (7.19) we obtain (7.18) and this completes the proof of Theorem 7.4.  $\square$

## APPENDIX A. CONTINUES ANALOGUES

This section is devoted to provide some vector-valued estimates in the continuous settings for maximal operators of Radon type. To fix notation let  $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{d_0}) : \mathbb{R}^k \rightarrow \mathbb{R}^{d_0}$  be a polynomial mapping whose components  $\mathcal{P}_j$  are real valued polynomials on  $\mathbb{R}^k$  such that  $\mathcal{P}_j(0) = 0$ . One of the main objects of our interest will be

$$\mathcal{M}_r^{\mathcal{P}} f(x) = \frac{1}{|B_r|} \int_{B_r} f(x - \mathcal{P}(y)) \, dy$$

where  $B_r$  is the Euclidean ball centered at the origin with radius  $r > 0$ . We prove the following.

**Theorem A.1.** *Assume that  $p \in (1, \infty)$  then there is a constant  $C_p > 0$  such that for every  $(f_t : t \in \mathbb{N}) \in L^p(\ell^2(\mathbb{R}^{d_0}))$  we have*

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{r > 0} |\mathcal{M}_r^{\mathcal{P}} f_t|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{L^p}.$$

Moreover, the implied constant is independent of the coefficients of the polynomial mapping  $\mathcal{P}$ .

Suppose that  $K \in \mathcal{C}^1(\mathbb{R}^k \setminus \{0\})$  is a Calderón–Zygmund kernel satisfying the differential inequality

$$(A.1) \quad |y|^k |K(y)| + |y|^{k+1} |\nabla K(y)| \leq 1$$

for all  $y \in \mathbb{R}^k \setminus \{0\}$  and the cancellation condition

$$(A.2) \quad \sup_{0 < r < R < \infty} \left| \int_{r \leq |y| \leq R} K(y) \, dy \right| \leq 1.$$

We will be also interested in truncated singular Radon transform

$$\mathcal{T}_r^{\mathcal{P}} f(x) = \int_{|y| > r} f(x - \mathcal{P}(y)) K(y) \, dy$$

for  $x \in \mathbb{R}^{d_0}$ .

The second main result will be the following.

**Theorem A.2.** *Assume that  $p \in (1, \infty)$  then there is a constant  $C_p > 0$  such that for every  $(f_t : t \in \mathbb{N}) \in L^p(\ell^2(\mathbb{R}^{d_0}))$  we have*

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{r > 0} |\mathcal{T}_r^{\mathcal{P}} f_t|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{L^p}.$$

Moreover, the implied constant is independent of the coefficients of the polynomial mapping  $\mathcal{P}$ .

We set

$$N_0 = \max\{\deg \mathcal{P}_j : 1 \leq j \leq d_0\}.$$

It is convenient to work with the set

$$\Gamma = \{\gamma \in \mathbb{Z}^k \setminus \{0\} : 0 \leq \gamma_j \leq N_0 \text{ for each } j = 1, \dots, k\}$$

with the lexicographic order. Then each  $\mathcal{P}_j$  can be expressed as

$$\mathcal{P}_j(x) = \sum_{\gamma \in \Gamma} c_j^\gamma x^\gamma$$

for some  $c_j^\gamma \in \mathbb{R}$ . Let us denote by  $d$  the cardinality of the set  $\Gamma$ . We identify  $\mathbb{R}^d$  with the space of all vectors whose coordinates are labeled by multi-indices  $\gamma \in \Gamma$ . Let  $A$  be a diagonal  $d \times d$  matrix such that

$$(Av)_\gamma = |\gamma| v_\gamma.$$

For  $t > 0$  we set

$$t^A = \exp(A \log t)$$

i.e.  $t^A x = (t^{|\gamma|} x_\gamma : \gamma \in \Gamma)$  for every  $x \in \mathbb{R}^d$ . Next, we introduce the *canonical* polynomial mapping

$$\mathcal{Q} = (\mathcal{Q}_\gamma : \gamma \in \Gamma) : \mathbb{Z}^k \rightarrow \mathbb{Z}^d$$

where  $\mathcal{Q}_\gamma(x) = x^\gamma$  and  $x^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$ . The coefficients  $(c_j^\gamma : \gamma \in \Gamma, j \in \{1, \dots, d_0\})$  define a linear transformation  $L : \mathbb{R}^d \rightarrow \mathbb{R}^{d_0}$  such that  $L\mathcal{Q} = \mathcal{P}$ . Indeed, it is enough to set

$$(Lv)_j = \sum_{\gamma \in \Gamma} c_j^\gamma v_\gamma$$

for each  $j \in \{1, \dots, d_0\}$  and  $v \in \mathbb{R}^d$ . Now in view of [7] or [15, Section 11] we can reduce the matters to the canonical polynomial mapping. Thus for simplicity of notation we will write  $\mathcal{M}_r = \mathcal{M}_r^{\mathcal{Q}}$  and  $\mathcal{T}_t = \mathcal{T}_t^{\mathcal{Q}}$ .

**A.1. Proof of Theorem A.1.** It suffices to show that for every  $p \in (1, \infty)$  there is a constant  $C_p > 0$  such that for every  $(f_t : t \in \mathbb{N}) \in L^p(\ell^2(\mathbb{R}^{d_0}))$  we have

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{Z}} |\mathcal{M}_{2^n} f_t|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{L^p}.$$

For this purpose let  $\Phi \in \mathcal{C}^\infty(\mathbb{R}^d)$  with compact support such that

$$\int_{\mathbb{R}^d} \Phi(y) \, dy = 1$$

and define for any  $r > 0$

$$\mathfrak{M}_r f(x) = \Phi_r * f(x)$$

where  $\Phi_r(y) = r^{-\text{tr}(A)}\Phi(r^{-A}y)$ . Then it is known from the classical vector-valued maximal estimates (see [15]) that

$$(A.3) \quad \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{Z}} |\mathfrak{M}_{2^n} f_t|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{L^p}.$$

Thus thanks to (A.3) it remains to prove that

$$(A.4) \quad \left\| \left( \sum_{t \in \mathbb{N}} \sum_{n \in \mathbb{Z}} |(\mathcal{M}_{2^n} - \mathfrak{M}_{2^n}) f_t|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{L^p}.$$

Appealing to Lemma 4.2 the proof of (A.4) will be completed if we show that for every  $p \in (1, \infty)$  and  $f \in L^p(\mathbb{R}^d)$  we have

$$(A.5) \quad \left\| \sum_{n \in \mathbb{Z}} \varepsilon_n(\omega) (\mathcal{M}_{2^n} - \mathfrak{M}_{2^n}) f \right\|_{L^p} \leq C_p \|f\|_{L^p}$$

for every  $(\varepsilon_n(\omega) : n \in \mathbb{N}_0)$  where  $\omega \in [0, 1]$  and  $\varepsilon_n(\omega) \in \{-1, 1\}$ .

In order to prove (A.5) we can assume without of loss of generality that  $p \geq 2$ . Define  $U_n = \varepsilon_n(\omega) (\mathcal{M}_{2^n} - \mathfrak{M}_{2^n})$ . Let  $S_j$  be a Littlewood–Paley projection  $\mathcal{F}(S_j g)(\xi) = \phi_j(\xi) \mathcal{F}g(\xi)$  associated with  $(\phi_j : j \in \mathbb{Z})$  a smooth partition of unity of  $\mathbb{R}^d \setminus \{0\}$  such that for each  $j \in \mathbb{Z}$  we have  $0 \leq \phi_j \leq 1$  and

$$\text{supp } \phi_j \subseteq \{\xi \in \mathbb{R}^d : 2^{-j-1} < |\xi| < 2^{-j+1}\}$$

and for  $\xi \in \mathbb{R}^d \setminus \{0\}$

$$\sum_{j \in \mathbb{Z}} \phi_j(\xi)^2 = 1.$$

Note that

$$\left\| \sum_{n \in \mathbb{Z}} \varepsilon_n(\omega) (\mathcal{M}_{2^n} - \mathfrak{M}_{2^n}) f \right\|_{L^p} \leq \sum_{j \in \mathbb{Z}} \left\| \sum_{n \in \mathbb{Z}} S_{j+n} U_n S_{j+n} f \right\|_{L^p} \lesssim \sum_{j \in \mathbb{Z}} \left\| \left( \sum_{n \in \mathbb{Z}} |U_n S_{j+n} f|^2 \right)^{1/2} \right\|_{L^p}.$$

Now we show that there is a constant  $\delta_p > 0$  such that

$$(A.6) \quad \left\| \left( \sum_{n \in \mathbb{Z}} |U_n S_{j+n} f|^2 \right)^{1/2} \right\|_{L^p} \lesssim 2^{-\delta_p |j|} \|f\|_{L^p}.$$

We set  $\mathcal{M}_* f = \sup_{r>0} |\mathcal{M}_r^* f|$  and  $\mathfrak{M}_* f = \sup_{r>0} |\mathfrak{M}_r^* f|$ , where  $\mathcal{M}_r^*$  and  $\mathfrak{M}_r^*$  is the adjoint operator to  $\mathcal{M}_r$  and  $\mathfrak{M}_r$  respectively. Now let  $g \in L^q(\mathbb{R}^d)$  such that  $\|g\|_{L^q} \leq 1$  and  $g \geq 0$ , where  $q = (p/2)'$ , and observe that by the Cauchy–Schwarz inequality

$$\begin{aligned} \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} |U_n S_{j+n} f(x)|^2 g(x) \, dx &\leq \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} |S_{j+n} f(x)|^2 (\mathcal{M}_* + \mathfrak{M}_*) g(x) \, dx \\ &\leq \left\| \left( \sum_{n \in \mathbb{Z}} |S_{j+n} f|^2 \right)^{1/2} \right\|_{L^p}^2 \|(\mathcal{M}_* + \mathfrak{M}_*) g\|_{L^q} \lesssim \|f\|_{L^p}^2 \|g\|_{L^q} \end{aligned}$$

since for all  $1 < q \leq \infty$  there is a constant  $C_q > 0$  such that

$$\|\mathcal{M}_* g\|_{L^q} + \|\mathfrak{M}_* g\|_{L^q} \leq C_q \|g\|_{L^q}$$

for all  $g \in L^p(\mathbb{R}^d)$ . Thus we have proven that

$$(A.7) \quad \left\| \left( \sum_{n \in \mathbb{Z}} |U_n S_{j+n} f|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}.$$

Now we refine the estimate (A.7) and we show that there is  $\delta > 0$  such that

$$(A.8) \quad \left\| \left( \sum_{n \in \mathbb{Z}} |U_n S_{j+n} f|^2 \right)^{1/2} \right\|_{L^2} \lesssim 2^{-\delta |j|} \|f\|_{L^2}.$$

Then interpolation of (A.7) with (A.8) will imply (A.6) and the proof of Theorem A.1 will be completed. Let

$$m_{2^n}(\xi) = |B_1|^{-1} \int_{B_1} e^{2\pi i \xi \cdot \mathcal{Q}(2^n y)} \, dy \quad \text{and} \quad \mathfrak{m}_{2^n}(\xi) = \mathcal{F}\Phi(2^n A \xi)$$



be the multipliers associated with the averages  $\mathcal{M}_{2^n}$  and  $\mathfrak{M}_{2^n}$  respectively. Then one sees that

$$|m_{2^n}(\xi) - \mathfrak{m}_{2^n}(\xi)| \lesssim \min \{ |2^{nA}\xi|_\infty, |2^{nA}\xi|_\infty^{-1/d} \}.$$

Hence, by Plancherel's theorem and the assumption on the supports for  $\phi_{n+j}$  we obtain

$$\begin{aligned} \left\| \left( \sum_{n \in \mathbb{Z}} |U_n S_{j+n} f|^2 \right)^{1/2} \right\|_{L^2}^2 &= \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} |m_{2^n}(\xi) - \mathfrak{m}_{2^n}(\xi)|^2 |\phi_{j+n}(\xi)|^2 |\mathcal{F}f(\xi)|^2 d\xi \\ &\lesssim 2^{-2\delta|j|} \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} |\phi_{j+n}(\xi)|^2 |\mathcal{F}f(\xi)|^2 d\xi \lesssim 2^{-2\delta|j|} \|f\|_{L^2}^2 \end{aligned}$$

as desired.

**A.2. Proof of Theorem A.2.** For each kernel  $K$  satisfying (A.1) and (A.2) there are functions  $(K_j : j \in \mathbb{Z})$  and a constant  $C > 0$  such that for  $y \neq 0$

$$(A.9) \quad K(y) = \sum_{j \in \mathbb{Z}} K_j(y),$$

where for each  $j \in \mathbb{Z}$  the kernel  $K_j$  is supported inside  $2^{j-2} \leq |y| \leq 2^j$ , satisfies

$$|y|^k |K_j(y)| + |y|^{k+1} |\nabla K_j(y)| \leq C$$

for all  $y \in \mathbb{R}^k$ , and has integral 0. We refer to [15, Chapter 6, §4.5, Chapter 13, §5.3] for more details.

Thanks to the decomposition (A.9) we have the pointwise bound for any  $f \geq 0$

$$(A.10) \quad \sup_{r>0} |\mathcal{T}_r f(x)| \lesssim \sup_{r>0} \mathcal{M}_r f(x) + \sup_{n \in \mathbb{Z}} \left| \sum_{j \geq n} T_j f(x) \right|$$

where

$$T_j f(x) = \int_{\mathbb{R}^k} f(x - \mathcal{Q}(y)) K_j(y) dy = \mu_{2^j} * f(x).$$

Theorem A.1 and (A.10) reduce the matters to proving that for every  $p \in (1, \infty)$  there exists a constant  $C_p > 0$  such that for every  $(f_t : t \in \mathbb{N}) \in L^p(\ell^2(\mathbb{R}^{d_0}))$  we have

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{Z}} \left| \sum_{j \geq n} T_j f_t \right|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{L^p}.$$

Let

$$Tf(x) = \sum_{j \in \mathbb{Z}} T_j f(x)$$

we know from [15] that

$$\|Tf\|_{L^p} \lesssim \|f\|_{L^p}.$$

Let  $\Phi \in \mathcal{C}^\infty(\mathbb{R}^d)$  with compact support such that

$$\int_{\mathbb{R}^d} \Phi(y) dy = 1.$$

As in [8] we decompose

$$\begin{aligned} T_n f &= \Phi_{2^n} * (Tf - \sum_{j < n} T_j f) + (\delta_0 - \Phi_{2^n}) * \sum_{j \geq n} T_j f \\ &= \Phi_{2^n} * Tf - (\Phi_{2^n} * \sum_{j < n} \mu_{2^j}) * f + \sum_{j \geq 0} (\delta_0 - \Phi_{2^n}) * \mu_{2^{j+n}} * f. \end{aligned}$$

We now observe that by (A.3) and (4.6) we have

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{Z}} |\Phi_{2^n} * T f_t|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{t \in \mathbb{N}} |T f_t|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{L^p}.$$

The function  $\Phi_{2^n} * \sum_{j < n} \mu_{2^j}$  defines a Schwartz function and for every  $N \in \mathbb{N}_0$

$$|\Phi_{2^n} * \sum_{j < n} \mu_{2^j}(x)| \lesssim_N 2^{-\text{tr}(A)n} (1 + |2^{nA}x|^2)^{-N}$$

thus by the classical vector-valued estimates [15] we get

$$\left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{Z}} \left| \left( \Phi_{2^n} * \sum_{j < n} \mu_{2^j} \right) * f_t \right|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{L^p}.$$

It remains to prove that there exists  $\delta_p > 0$  such that

$$(A.11) \quad \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{Z}} \left| (\delta_0 - \Phi_{2^n}) * \mu_{2^{j+n}} * f_t \right|^2 \right)^{1/2} \right\|_{L^p} \lesssim 2^{-\delta_p j} \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{L^p}.$$

For  $p \in (1, \infty)$ , due to Theorem A.1 and (A.3) we have

$$(A.12) \quad \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{Z}} \left| (\delta_0 - \Phi_{2^n}) * \mu_{2^{j+n}} * f_t \right|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{L^p}.$$

For  $p = 2$  we show that there exists  $\delta > 0$  such that

$$(A.13) \quad \left\| \left( \sum_{t \in \mathbb{N}} \sup_{n \in \mathbb{Z}} \left| (\delta_0 - \Phi_{2^n}) * \mu_{2^{j+n}} * f_t \right|^2 \right)^{1/2} \right\|_{L^2} \lesssim 2^{-\delta j} \left\| \left( \sum_{t \in \mathbb{N}} |f_t|^2 \right)^{1/2} \right\|_{L^2}.$$

Then interpolation (A.12) with (A.13) will establish (A.11) and the proof of Theorem A.2 will be completed. It suffices to show

$$\left\| \sup_{n \in \mathbb{Z}} \left| (\delta_0 - \Phi_{2^n}) * \mu_{2^{j+n}} * f \right| \right\|_{L^2} \lesssim 2^{-\delta j} \|f\|_{L^2}.$$

By Plancherel's theorem we see

$$\begin{aligned} \left\| \sup_{n \in \mathbb{Z}} \left| (\delta_0 - \Phi_{2^n}) * \mu_{2^{j+n}} * f \right| \right\|_{L^2}^2 &\leq \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} \left| (1 - \mathcal{F}\Phi(2^{nA}\xi)) \mathcal{F}\mu_{2^{j+n}}(\xi) \right|^2 |\mathcal{F}f(\xi)|^2 d\xi \\ &\lesssim \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} |2^{nA}\xi|^{1/d} |2^{(j+n)A}\xi|^{-1/d} \min \{ |2^{(j+n)A}\xi|_\infty, |2^{(j+n)A}\xi|_\infty^{-1/d} \} |\mathcal{F}f(\xi)|^2 d\xi \\ &\lesssim 2^{-j/d} \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} \min \{ |2^{(j+n)A}\xi|_\infty, |2^{(j+n)A}\xi|_\infty^{-1/d} \} |\mathcal{F}f(\xi)|^2 d\xi \lesssim 2^{-j/d} \|f\|_{L^2}^2 \end{aligned}$$

as claimed, since

$$|1 - \mathcal{F}\Phi(2^{nA}\xi)| \lesssim |2^{nA}\xi|_\infty^{1/d}$$

and

$$|\mathcal{F}\mu_{2^{j+n}}(\xi)| \lesssim \min \{ |2^{(j+n)A}\xi|_\infty, |2^{(j+n)A}\xi|_\infty^{-1/d} \}.$$

The proof of Theorem A.2 is completed.

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